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Three lectures on quantum groups: representations, duality, real forms

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In these lectures three topics are discussed: the representations of quantum groups, duality between quantum algebras and matrix quantum groups and q-deformations of real forms of quantum groups.

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Contents

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Introduction

Quantum groups appeared first as *quantum algebras*, i.e., as one-parameter deformations of the universal enveloping algebras of complex simple Lie algebras, in the study of the algebraic aspects of quantum integrable systems in the papers of Faddeev, Kulish, Reshetikhin and Sklyanin [F, KR1, KS, KRS, S1, S2]. For recent reviews we refer to refs. [FT2, FRT]. Then quantum algebras related to trigonometric solutions of the quantum Yang–Baxter equation were axiomatically introduced as (pseudo) quasi-triangular Hopf algebras independently by Drinfeld and Jimbo [D1, J1, J2, D2]. The mathematics of these objects was studied in refs. [R1, R2, R3, L1, DK] (for more references on the representation theory of quantum algebras cf. refs. [Do1–Do4]). Later, inspired by the Knizhnik–Zamolodchikov equations Drinfeld developed a theory of formal deformations and introduced a new notion of quasi-Hopf algebras [D3].

Other approaches to quantum groups, in which the objects may be called *quantum matrix groups* and are Hopf algebras in duality to the quantum algebras, are developed by Faddeev, Reshetikhin and Takhtajan [FRT], Manin [M1, M2] and Woronowicz. The first approach, called R-matrix approach, is based on the main relation of the quantum inverse scattering method. There the quantum group matrices M played the role of quantum monodromy matrices (with operator-valued entries) of the auxiliary linear problem and the Yang–Baxter equation was a compatibility equation. The approach of Manin considers quantum groups which act as symmetries of non-commutative, or quantum, spaces. The resulting objects are the same as in the first approach. For the approach of Woronowicz we refer to ref. [W]. For the connections between the different approaches we refer the reader to refs. [R1, Mj, DHL]. We should mention also the development by Wess, Zumino and collaborators of differential calculus on quantum hyperplanes (cf., e.g., refs. [WZ, SWZ]).

Because of the lack of space these notes represent only the lectures given by the author at the School. They can be viewed as an update of two earlier long reviews [Do2, Do3]. Nevertheless, these lectures are self-contained.

1. Preliminaries

1.1. HOPF ALGEBRAS AND QUANTUM GROUPS

Let F be a field of characteristic 0; in fact most of the time we shall work with $F=\mathbb{C}$ or $F=\mathbb{R}$. An associative algebra $\mathcal{U}$ over F with unity $1_{\mathcal{U}}$ is called a *bialgebra* [A] if there exist two algebra homomorphisms called *co-multiplication* δ:

$$\delta: \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}\,, \quad \delta(1_{\mathcal{U}}) = 1_{\mathcal{U}} \otimes 1_{\mathcal{U}}\,, \qquad (1.1a)$$

and *co-unit* ϵ:

$$\epsilon: \mathcal{U} \to F, \quad \epsilon(1_{\mathcal{U}}) = 1 . \tag{1.1b}$$

The co-multiplication δ fulfills the axiom of co-associativity:

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta , \tag{1.1c}$$

where both sides are maps $\mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$; the two homomorphisms fulfil

$$(\mathrm{id} \otimes \epsilon) \circ \delta = i_1 , \qquad (\epsilon \otimes \mathrm{id}) \circ \delta = i_2 , \tag{1.1d}$$

as maps $\mathcal{U} \to F \otimes \mathcal{U}$ and $\mathcal{U} \to \mathcal{U} \otimes F$, respectively, where i_1, i_2 are the maps identifying $\mathcal{U}$ with $\mathcal{U} \otimes F$ and $F \otimes \mathcal{U}$, respectively. Next a bialgebra $\mathcal{U}$ is called a *Hopf algebra* [A] if there exists an algebra antihomomorphism γ called *antipode*:

$$\gamma: \mathcal{U} \to \mathcal{U} , \quad \gamma(1_{\mathcal{U}}) = 1_{\mathcal{U}} , \tag{1.2a}$$

such that the following axiom is fulfilled:

$$m \circ (\mathrm{id} \otimes \gamma) \circ \delta = i \circ \epsilon , \tag{1.2b}$$

as maps $\mathcal{U} \to \mathcal{U}$, where m is the usual product in the algebra: $m(Y \otimes Z) = YZ$, Y, $Z \in \mathcal{U}$ and i is the natural embedding of F into $\mathcal{U}: i(c) = c1_{\mathcal{U}}$, $c \in F$. The antipode plays the role of an inverse although there is no requirement that $\gamma^2 = \mathrm{id}$.

One needs also the *opposite co-multiplication* [J2, D2] $\delta' = \pi \circ \delta$, where π is the permutation in $\mathcal{U} \otimes \mathcal{U}$. If the antipode has an inverse then one uses also the notion of opposite antipode [J2, D2]: $\gamma' = \gamma^{-1}$.

A Hopf algebra $\mathcal{U}$ is called a *quasi-triangular Hopf algebra* or *quantum group* [D2] if there exists an invertible element $R \in \mathcal{U} \otimes \mathcal{U}$, called universal R-matrix [D1, D2], which intertwines δ and δ':

$$R\, \delta(Y) = \delta'(Y)\, R , \quad \forall Y \in \mathcal{U} , \tag{1.3a}$$

and obeys also the relations

$$(\delta \otimes \mathrm{id}) R = R_{13} R_{23} , \quad R = R_{\cdot 3} , \tag{1.3b}$$

$$(\mathrm{id} \otimes \delta) R = R_{13} R_{12} , \quad R = R_{1\cdot} , \tag{1.3c}$$

where the indices indicate the embeddings of R into $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$. From the above it follows that

$$(\epsilon \otimes \mathrm{id}) R = (\mathrm{id} \otimes \epsilon) R = 1_{\mathcal{U}} , \tag{1.4a}$$

and using also (1.4a) one has

$$(\gamma \otimes \mathrm{id}) R = R^{-1} , \qquad (\mathrm{id} \otimes \gamma) R^{-1} = R . \tag{1.4b}$$

The term *quantum group* is used [D2] also if R is not in $\mathcal{U} \otimes \mathcal{U}$ but in some completion of it (cf. next subsection).

From (1.3a) and one of (1.3b,c) follows the Yang–Baxter equation for R:

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12} . \tag{1.5}$$

A quasi-triangular Hopf algebra is called a *triangular Hopf algebra* if also the following holds:

$$\pi R^{-1}=R . \tag{1.6}$$

1.2. QUANTUM ALGEBRAS

From now on (unless specified otherwise) we set $F=\mathbb{C}$. Let $\mathcal{G}$ be a complex simple Lie algebra; then the q-deformation $U_q(\mathcal{G})$ of the universal enveloping algebras $U(\mathcal{G})$ is defined [D1, J1, J2, D2] as the associative algebra over $\mathbb{C}$ with generators $X_i^\pm$, H_i, $i=1, ..., l=\text{rank } \mathcal{G}$ and with relations

$$[H_i, H_j]=0 , \qquad [H_i, X_j^\pm]=\pm a_{ij}X_j^\pm , \tag{1.7}$$

$$[X_i^+, X_j^-]=\delta_{ij}\frac{q_i^{H_i/2}-q_i^{-H_i/2}}{q_i^{1/2}-q_i^{-1/2}}=\delta_{ij}[H_i]_{q_i}, \quad q_i=q^{(\alpha_i,\alpha_i)/2} , \tag{1.8}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}_{q_i} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{n-k}=0 , \quad i\neq j , \tag{1.9}$$

where $(a_{ij})=(2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))$ is the Cartan matrix of $\mathcal{G}$, $(\cdot , \cdot)$ is the scalar product of the roots normalized so that for the short roots α we have $(\alpha, \alpha)=2$, $n=1-a_{ij}$,

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!}, \quad [m]_q!=[m]_q[m-1]_q\cdots[1]_q , \tag{1.10a}$$

$$[m]_q=\frac{q^{m/2}-q^{-m/2}}{q^{1/2}-q^{-1/2}}=\frac{\sinh(mh/2)}{\sinh(h/2)}=\frac{\sin(\pi m\tau)}{\sin(\pi\tau)} ,$$

$$q=e^h=e^{2\pi i\tau} , \quad h, \tau\in\mathbb{C} , \tag{1.10b}$$

$$q_i^{a_{ij}}=q^{(\alpha_i,\alpha_j)}=q_j^{a_{ji}} . \tag{1.10c}$$

Further we shall omit the subscript q in $[m]_q$ if no confusion can arise. Note also that instead of q some authors use $q'=q^2$. This definition is valid also when $\mathcal{G}$ is an affine Kac–Moody algebra [D1].

The algebras $U_q(\mathcal{G})$ were called *quantum groups* [D1, D2], or *quantum universal enveloping algebras* [Re1, KiR1]. For shortness we shall call them *quantum algebras* as is becoming commonly accepted in the recent literature.

For $q\to 1$ $(h\to 0)$, we recover the standard commutation relations from (1.7), (1.8) and Serre's relations from (1.9) in terms of the Chevalley generators H_i, $X_i^\pm$. The elements H_i span the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$, while the elements $X_i^\pm$ generate the subalgebras $\mathcal{G}^\pm$. We shall use the standard decompositions into direct sums of vector subspaces

$$\mathscr{G}=\mathscr{H}\oplus\bigoplus_{\beta\in\Delta}\mathscr{G}_\beta=\mathscr{G}^+\oplus\mathscr{H}\oplus\mathscr{G}^-\,,\quad \mathscr{G}^\pm=\bigoplus_{\beta\in\Delta^\pm}\mathscr{G}_\beta\,,$$

where $\Delta=\Delta^+\cup\Delta^-$ is the root system of $\mathscr{G}$, and Δ^+ and Δ^- are the sets of positive and negative roots, respectively; Δ_S will denote the set of simple roots of Δ. We recall that H_j correspond to the simple roots α_j of $\mathscr{G}$, and if $\beta^\vee=\sum_j n_j\alpha_j^\vee$, $\beta^\vee\equiv 2\beta/(\beta,\beta)$, then to β there corresponds $H_\beta=\sum_j n_jH_j$. The elements of $\mathscr{G}$ which span $\mathscr{G}_\beta$ ($\dim\mathscr{G}_\beta=1$) are denoted by X_β. These Cartan–Weyl generators H_β, X_β [J1, J2, Do1, T1] may be normalized so that

$$[X_\beta,X_{-\beta}]=[H_\beta]_{q_\beta}\,,\qquad [H_\beta,X_{\pm\beta'}]=\pm(\beta^\vee,\beta')X_{\pm\beta'}\,,$$
$$\beta,\beta'\in\Delta^+\,,\quad q_\beta\equiv q^{(\beta,\beta)/2}\,. \tag{1.11}$$

In some considerations it is necessary to use a subalgebra $\tilde{U}_q(\mathscr{G})$ of $U_q(\mathscr{G})$ generated by $X_i^\pm$ and

$$K_i^{\pm1}=q_i^{\pm H_i/4}\,; \tag{1.12}$$

then (1.7) and (1.8) are replaced by

$$K_iK_i^{-1}=K_i^{-1}K_i=1\,,\qquad [K_i,K_j]=0\,,\qquad K_i,X_j^\pm K_i^{-1}=q_i^{\pm a_{ij}/4}X_j^\pm\,, \tag{1.7'}$$

$$[X_i^+,X_j^-]=\delta_{ij}\frac{K_i^2-K_i^{-2}}{q_i^{1/2}-q_i^{-1/2}}\,. \tag{1.8'}$$

One may also use instead of $X_i^\pm$ the generators [R3]

$$E_i=X_i^+q_i^{-H_i/4}=X_i^+K_i^{-1}\,,\qquad F_i=X_i^-q_i^{H_i/4}=X_i^-K_i\,. \tag{1.13}$$

In ref. [S2] for $\mathscr{G}=\mathrm{sl}(2,\mathbb{C})$ and in refs. [D1, J1, J2, D2] in general it was observed that the algebra $U_q(\mathscr{G})$ is a Hopf algebra, the co-multiplication, co-unit, and antipode being defined on the generators of $U_q(\mathscr{G})$ as follows:

$$\delta(H_i)=H_i\otimes1+1\otimes H_i\,,\qquad \delta(X_i^\pm)=X_i^\pm\otimes q_i^{H_i/4}+q_i^{-H_i/4}\otimes X_i^\pm\,, \tag{1.14a}$$

$$\epsilon(H_i)=\epsilon(X_i^\pm)=0\,, \tag{1.14b}$$

$$\gamma(H_i)=-H_i\,,\qquad \gamma(X_i^\pm)=-q_i^{\hat\rho/2}X_i^\pm q_i^{-\hat\rho/2}=-q_i^{\pm1/2}X_i^\pm\,, \tag{1.14c}$$

where $\hat\rho\in\mathscr{H}$ corresponds to $\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$, Δ^+ is the set of positive roots, $\hat\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}H_\alpha$. The action of δ, ϵ, γ on the Cartan–Weyl generators is obtained easily from (1.14) since H_β and X_β are given algebraically in terms of the Chevalley generators. [Note that if $\alpha\notin\Delta_S$ the co-algebra operations δ, γ look more complicated than (1.14).] The axioms in (1.1), (1.2) are fulfilled by the explicit definition (1.14). The opposite co-multiplication and antipode [J2, D2] introduced above define a Hopf algebra $U_q(\mathscr{G})'$ which is related to $U_q(\mathscr{G})$ by

$$U_q(\mathscr{G})'=U_{q^{-1}}(\mathscr{G})\,. \tag{1.15}$$

In terms of the generators $K_i^{\pm 1}$, E_i, F_i the above relations are rewritten as follows:

$$\delta(K_i)=K_i\otimes K_i\,,\qquad \delta(E_i)=E_i\otimes 1+K_i^{-2}\otimes E_i\,,$$

$$\delta(F_i)=F_i\otimes K_i^2+1\otimes F_i\,, \tag{1.14a$'$}$$

$$\epsilon(K_i)=1\,,\qquad \epsilon(E_i)=\epsilon(F_i)=0\,, \tag{1.14b$'$}$$

$$\gamma(K_i)=K_i^{-1}\,,\qquad \gamma(E_i)=-K_i^2E_i\,,\qquad \gamma(F_i)=-F_iK_i^{-2}\,. \tag{1.14c$'$}$$

One may also rewrite the Serre relations (1.9) as [R3]:

$$(\mathrm{ad}_qE_i)^n(E_j)=0=(\mathrm{ad}'_qF_i)^n(F_j)\,,\quad i\neq j\,, \tag{1.9$'$a}$$

where

$$\mathrm{ad}_q:U_q(\mathscr{G}^+)\to\mathrm{End}(U_q(\mathscr{G}^+))\,,\quad \mathrm{ad}_q=(L\otimes R)(\mathrm{Id}\otimes\gamma)\delta\,, \tag{1.9$'$b}$$

$$\mathrm{ad}'_q:U_q(\mathscr{G}^-)\to\mathrm{End}(U_q(\mathscr{G}^-))\,,\quad \mathrm{ad}'_q=(L\otimes R)(\mathrm{Id}\otimes\gamma')\delta'\,, \tag{1.9$'$c}$$

and L (R) is the left (right) representation. Furthermore $\mathrm{ad}_q(E_i)$ acts as a twisted derivation: for $X,Y\in U_q(\mathscr{G}^+)$ homogeneous of degree β, $\gamma\in\mathscr{H}^*$ we have

$$\mathrm{ad}_q(E_i)(XY)=\mathrm{ad}_q(E_i)(X)Y+q^{-(\alpha_i,\gamma)/2}X\,\mathrm{ad}_q(E_i)(Y)\,.$$

The action of $\mathrm{ad}'_q(F_i)$ on X, $Y\in U_q(\mathscr{G}^-)$ is defined analogously.

For $\mathscr{G}=\mathrm{sl}(2,\mathbb{C})$ the universal R-matrix is given explicitly by [D2]

$$R=q^{H\otimes H/4}\sum_{n\geq 0}\frac{(1-q^{-1})^nq^{n(n-1)/4}}{[n]!}(q^{H/4}X^+)^n\otimes(q^{-H/4}X^-)^n\,, \tag{1.16}$$

where $H=H_1$, $X^\pm=X_1^\pm$, $r=1$.

Note that this R-matrix is not in $U_q(\mathrm{sl}(2,\mathbb{C}))\otimes U_q(\mathrm{sl}(2,\mathbb{C}))$, since it contains power series involving the generators $X^\pm$, but in some completion of it [in the h-adic topology used in refs. [D1, D2] ($q=\mathrm{e}^h$)]. This is valid for the R-matrices of all $U_q(\mathscr{G})$. Hopf algebras with such an R-matrix are called *pseudo quasi-triangular Hopf algebras* [D2] or *essentially quasi-triangular Hopf algebras* [Mj]. For $\mathscr{G}=\mathrm{sl}(n,\mathbb{C})$ an explicit formula for R was given in ref. [R3]. Then explicit multiplicative formulas for R were given in refs. [KiR2, LS] for all complex simple Lie algebras $\mathscr{G}$ and in ref. [KT] for all finite-dimensional superalgebras with symmetrizable Cartan matrices.

The centre of $U_q(\mathscr{G})$ and for generic q the centre of $\tilde{U}_q(\mathscr{G})$ is generated by q-analogues of the Casimir operators [S2, J1, J2]. For $\mathscr{G}=\mathrm{sl}(2,\mathbb{C})$ one has

$$C_2=[(H+1)/2]^2+X^-X^+\,. \tag{1.17}$$

For $U_q(\mathrm{sl}(n+1,\mathbb{C}))$ the (second-order) Casimir operator was given in ref. [MZ].

1.3. MATRIX QUANTUM GROUPS

The *quantum plane* [M1] $R_q(n|0)$, or rather the polynomial ring on it, is generated by coordinates x_i, $i=1,\ldots,n$, with commutation rules

$$x_i x_j = q^{1/2} x_j x_i \,, \quad \text{for } i<j \,. \tag{1.18a}$$

The *Grassmannian quantum plane* [M1] $R_q(0|n)$ is generated by coordinates ξ_i, $i=1, ..., n$, which satisfy

$$\xi_i^2 = 0 \,, \qquad \xi_i \xi_j = -q^{-1/2} \xi_j \xi_i \,, \quad \text{for } i<j \,. \tag{1.18b}$$

Consider next $n\times n$ matrices M with non-commuting matrix elements, or *quantum matrices*, which perform linear transformations of $R_q(n|0)$ and $R_q(0|n)$:

$$\{x'_1, ..., x'_n\} \in R_q(n|0) \,, \quad x'_i = M_{ij} x_j \,, \tag{1.19a}$$

$$\{\xi'_1, ..., \xi'_n\} \in R_q(0|n) \,, \quad \xi'_i = M_{ij} \xi_j \,, \tag{1.19b}$$

where one assumes that the elements of M commute with all x_i, ξ_i. Implementation of (1.19) gives the following restrictions upon the elements of M:

$$M_{ij} M_{il} = q^{1/2} M_{il} M_{ij} \,, \qquad \text{for } j<l \,, \tag{1.20a}$$

$$M_{ij} M_{kj} = q^{1/2} M_{kj} M_{ij} \,, \qquad \text{for } i<k \,, \tag{1.20b}$$

$$M_{il} M_{kj} = M_{kj} M_{il} \,, \qquad \text{for } i<k, j<l \,, \tag{1.20c}$$

$$[M_{ij}, M_{kl}] = (q^{1/2} - q^{-1/2}) M_{il} M_{kj} \,, \quad \text{for } i<k, j<l \,. \tag{1.20d}$$

Let us denote by $A_q(n)$ the bialgebra generated by the matrix elements M_{ij}, $i, j=1, ..., n$, with the following co-multiplication δ and co-unit ϵ:

$$\delta(M_{ij}) = \sum_{k=1}^{n} M_{ik} \otimes M_{kj} \,, \quad \text{or} \quad \delta(M) = M \hat{\otimes} M \,, \tag{1.21a}$$

$$\epsilon(M_{ij}) = \delta_{ij} \,, \quad \text{or} \quad \epsilon(M) = I_n \,, \tag{1.21b}$$

where $\hat{\otimes}$ denotes the tensor product of algebras and the usual product of matrices, I_n is the unit $n\times n$ matrix.

Further, a quantum determinant is defined in the following way:

$$\det_q M = \sum_{i_1, ..., i_n} M_{1,i_1} M_{1,i_2} \cdots M_{1,i_n} (-q^{1/2})^{l(i_1, ..., i_n)} \,, \tag{1.22}$$

where $l(i_1, ..., i_n)$ is the number of inversions in the permutation $(i_1, ..., i_n)$. Note that

$$\delta(\det_q M) = \det_q M \otimes \det_q M \,, \tag{1.23a}$$

$$\begin{aligned} \epsilon(\det_q M) &= \sum_{i_1, ..., i_n} \epsilon(M_{1,i_1}) \cdots \epsilon(M_{1,i_n}) (-q^{1/2})^{l(i_1, ..., i_n)} \\ &= \sum_{i_1, ..., i_n} \delta_{1,i_1} \cdots \delta_{1,i_n} (-q^{1/2})^{l(i_1, ..., i_n)} = 1 \,. \end{aligned} \tag{1.23b}$$

It is easy to check that $\det_q M$ commutes with the elements of M. Next let M',

$M'' \in A_q(n)$, and let all elements of M' commute with all elements of M''. Then both products $M'M''$, $M''M' \in A_q(n)$ and $\det_q M'M'' = \det_q M''M' = \det_q M' \det_q M'' = \det_q M'' \det_q M'$. If $\det_q M \neq 0$ then one can find a matrix M^{-1} which is both left and right inverse of M. However, M^{-1} belongs to $A_{q^{-1}}(n)$ instead of $A_q(n)$.

Thus one can obtain the quantum groups $GL_q(n, \mathbb{C})$ and $SL_q(n, \mathbb{C})$, as the Hopf algebras generated by the matrix elements M_{ij}, $i, j = 1, ..., n$, such that the condition $\det_q M \neq 0$ and $\det_q M = 1$, respectively, holds [FT2, M1, W, CFFS]. The antipode is given by the formula

$$\gamma(M) = M^{-1}, \qquad \gamma(\det_q M) = (\det_q M)^{-1}. \tag{1.24}$$

(Woronowicz [W] calls these objects also quantum pseudogroups.) The above notation is natural since for $q=1$ and assuming that M_{ij} become complex numbers one obtains the standard commutative Hopf algebras of polynomial functions on the classical groups $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ with co-multiplication and co-unit given by (1.21) and the antipode given by (1.24) with $q=1$. Of course in the $q=1$ case one works usually with the groups $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ themselves without reference to this related Hopf algebra (even when one considers tensor products of group representations which are by default governed by the co-multiplication structure).

The quantum group $SL_q(n, \mathbb{C})$ is dual to the quantum algebra $U_q(sl(n, \mathbb{C}))$. This duality is manifested in several forms. The first is through the R-matrices [cf. (1.12)]. The R-matrix of $U_q(sl(n, \mathbb{C}))$ in the fundamental representation has the form [FRT]

$$R_n = q^{1/2} \sum_{i=1}^{n} E_{ii} \hat{\otimes} E_{ii} + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} E_{ii} \hat{\otimes} E_{jj} + (q^{1/2} - q^{-1/2}) \sum_{\substack{i,j=1 \\ i > j}}^{n} E_{ij} \hat{\otimes} E_{ji}. \tag{1.25}$$

Now one may check that the following relation holds:

$$R_n M_1 M_2 = M_2 M_1 R_n, \tag{1.26}$$

where $M_1 = M \hat{\otimes} I_n$, $M_2 = I_n \hat{\otimes} M$.

Conversely, one may start with relation (1.26) imposing it on an arbitrary $n \times n$ matrix M; then one would obtain relations (1.20). This characterizes the approach of Faddeev, Reshetikhin and Takhtajan [FRT], for which the starting point is formula (1.26) and the Yang–Baxter equation (1.15). Their motivation comes from the original context of the quantum inverse scattering method [FST, FT1, F], where the matrix M played the role of quantum monodromy matrix (with operator-valued entries) of the auxiliary linear problem and the Yang–Baxter equation was a compatibility equation for eq. (1.26). Following their approach, Faddeev, Reshetikhin and Takhtajan [FRT] have defined in a similar way the quantum groups $SO_q(n, \mathbb{C})$, $Sp_q(n, \mathbb{C})$.

2. Representations of $U_q(\mathcal{G})$

2.1. VERMA MODULES AND THEIR IRREDUCIBLE SUBQUOTIENTS

The highest weight modules (HWM) V over $U_q(\mathcal{G})$ [J1] are given by their highest weight $\lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V$ such that

$$X_i^+ v_0 = 0\,, \quad i=1, ..., l\,, \qquad Hv_0 = \lambda(H) v_0\,, \quad H \in \mathcal{H}\,. \tag{2.1}$$

We define a *Verma module* V^λ as the HWM over $U_q(\mathcal{G})$ with highest weight $\lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V^\lambda$, induced from the one-dimensional representation $V_0 \cong \mathbb{C} v_0$ of $U_q(\mathcal{B})$, where $\mathcal{B} = \mathcal{B}^+$, $\mathcal{B}^\pm = \mathcal{H} \oplus \mathcal{G}^\pm$ are Borel subalgebras of $\mathcal{G}$, such that $U_q(\mathcal{G}^+) v_0 = 0$, $Hv_0 = \lambda(H) v_0$, $H \in \mathcal{H}$. (Note that the algebras $U_q(\mathcal{B}^\pm)$ with generators H_i, $X_i^\pm$ are Hopf subalgebras of $U_q(\mathcal{G})$ [Re1].) Thus one has

$$V^\lambda \cong U_q(\mathcal{G}) \otimes_{U_q(\mathcal{B})} v_0 \cong U_q(\mathcal{G}^-) \otimes v_0\,.$$

The representation theory of V^λ parallels the theory of Verma modules $V(\Lambda)$ over $\mathcal{G}$. [$V(\Lambda)$ is defined as the HWM over $\mathcal{G}$ induced from the one-dimensional representations of $\mathcal{B}$.]

We recall several facts from ref. [Do1]. The Verma module V^λ is reducible if there exists a root $\beta \in \Delta^+$ and $m \in \mathbb{N}$ such that

$$[(\lambda+\rho, \beta^\vee) - m]_{q\beta} = [(\lambda+\rho)(H_\beta) - m]_{q\beta} = 0\,, \quad \beta^\vee \equiv 2\beta/(\beta,\beta)\,, \tag{2.2}$$

holds. If q is not a root of unity then (2.2) is also a necessary condition for reducibility and then it may be rewritten as $2(\lambda+\rho, \beta) = m(\beta,\beta)$. [In that case it is the generalization of the (necessary and sufficient) reducibility conditions for Verma modules over finite-dimensional $\mathcal{G}$ and affine Lie algebras.] For uniformity we shall write the reducibility condition in the general form (2.2). If (2.2) holds then there exists a vector $v_s \in V^\lambda$, called a *singular vector*, such that

$$v_s \neq v_0\,, \qquad X_i^+ v_s = 0\,, \quad i=1, ..., l\,,$$
$$Hv_s = (\lambda(H) - m\beta(H)) v_s\,, \quad \forall H \in \mathcal{H}\,.$$

The space $U_q(\mathcal{G}^-) v_s$ is a proper submodule of V^λ isomorphic to the Verma module $V^{\lambda - m\beta} = U_q(\mathcal{G}^-) \otimes v_0'$, where v_0' is the highest weight vector of $V^{\lambda - m\beta}$, the isomorphism being realized by $v_s \mapsto 1 \otimes v_0'$. This situation will be denoted by $V^\lambda \to V^{\lambda - m\beta}$, i.e., we use the usual convention that the arrows depicting the embedding maps point *to* the embedded HWM. The singular vector is given by [Do1]

$$v_s = v^{\beta,m} = \mathcal{P}^\beta_m(X_1^-, ..., X_l^-) \otimes v_0\,, \tag{2.3}$$

where $\mathcal{P}^\beta_m$ is a homogeneous polynomial in its variables of degrees mn_i, where

$n_i \in \mathbb{Z}_+$ come from $\beta = \sum n_i \alpha_i$, α_i being the system of simple roots. The polynomial $\mathcal{P}_m^\beta$ is unique up to a non-zero multiplicative constant. If (2.2) holds for several pairs $(m, \beta) = (m_i, \beta_i)$, $i = 1, ..., k$, there are other HWM modules $V^{\lambda - m_i \beta_i}$ all of which are isomorphic to submodules of V^λ. The Verma module V^λ contains a unique proper maximal submodule I^λ. Among the HWM with highest weight λ there is a unique irreducible one, denoted by L_λ, i.e.,

$$L_\lambda = V^\lambda / I^\lambda . \tag{2.4}$$

If V^λ is irreducible then $L_\lambda = V^\lambda$.

Suppose that q is not a root of 1. Then the representations of $U_q(\mathcal{G})$ are deformations of the representations of $U(\mathcal{G})$, and the latter are obtained from the former for $q \to 1$ [R2, L1]. Consider V^λ reducible with respect to (w.r.t.) to every simple root (and thus w.r.t. to all positive roots):

$$[(\lambda + \rho, \alpha_i^\vee) - m_i]_{q_i} = [\lambda(H_i) + 1 - m_i]_{q_i} = 0 , \quad m_i \in \mathbb{N}, i = 1, ..., l , \tag{2.5}$$

where we used $\rho(\alpha_i^\vee) = 1$. Then L_λ is a finite-dimensional highest weight module over $U_q(\mathcal{G})$ and all such modules may be obtained in this way [KiR2, KR1, R2, L1]. If we restrict $U_q(\mathcal{G})$ to its compact real form $U_q(\mathcal{G}_k)$ then the set of all L_λ coincides with the set of all finite-dimensional unitary irreducible representations of $U_q(\mathcal{G}_k)$.

Recently, De Concini and Kac [DK] have given a formula for the determinant of the contravariant form on the Verma modules V^λ. This result implies in the usual way the description of irreducible subquotients of V^λ. In particular, this confirms results on the embeddings of the reducible modules V^λ [Do1] summarized partially above.

If the deformation parameter q is a root of unity the representation theory of $U_q(\mathcal{G})$ differs very much from the generic case (cf., e.g., refs. [L2, DK, Do1]). For the lack of space we do not discuss this case and we refer to the contributions of Felder and of Cuerno, Sierra and Gomez in these Proceedings (cf. also ref. [Do2]).

2.2. SINGULAR VECTORS

The importance of the singular vectors was explained in the previous subsection. In this subsection we present, following ref. [Do4], the singular vectors corresponding to a class of positive roots which we call straight roots. For lack of space we shall restrict ourselves to the case when q is not a root of 1; otherwise we refer to ref. [Do4].

In order to introduce this class we shall need the *Weyl group* W of the simple complex Lie algebra $\mathcal{G}$. The group W is generated by the *Weyl reflections* s_α, $\alpha \in \Delta$, of $\mathcal{H}^*$, defined by $s_\alpha(\lambda) \equiv \lambda - (\lambda, \alpha^\vee)\alpha$. Actually W is generated by the simple reflections $s_i \equiv s_{\alpha_i}$, $1 \leqslant i \leqslant l$, corresponding to the simple roots α_i. Thus every element $w \in W$ may be written as the product of some simple reflections. Every such product which uses a minimal number of simple reflections is called a *reduced expression* or *reduced form* for w. The number of simple reflections in the reduced form is called the *length* of w and is denoted by $l(w)$. Note that there may exist many reduced forms for a fixed w. We shall also need the *Weyl dot reflections* $w \cdot \lambda$ defined by $w \cdot \lambda \equiv w(\lambda+\rho)-\rho$. Note that if (2.2) is fulfilled and q is not a root of 1 then $\lambda - m\beta = s_\beta \cdot \lambda$.

It is well known [B] that every root may be expressed as the result of the action of an element of the Weyl group W on some simple root. More explicitly, for any $\beta \in \Delta^+$ we have:

$$\beta = w(\alpha_p) = s_{p_1} s_{p_2} \cdots s_{p_r}(\alpha_p) , \tag{2.6a}$$

and consequently

$$s_\beta = w s_p w^{-1} = s_{p_1} \cdots s_{p_r} s_p s_{p_r} \cdots s_{p_1} , \tag{2.6b}$$

where α_p is a simple root, and the element $w \in W$ is written in a reduced form. The positive root β is called a *straight root* if all numbers $p, p_1, p_2, \ldots, p_r$ in (2.6a) are different. Note that there may exist different forms of (2.6) involving other elements w' and $\alpha_{p'}$; however, this definition does not depend on the choice of these elements. Obviously, any simple root is a straight root. Other easy examples of straight roots are those which are sums of simple roots with coefficients not exceeding 1, i.e., $\beta = \sum_i n_i \alpha_i$, with $n_i = 1$ or 0. All straight roots of the simply laced algebras A_l, D_l, E_l are of this form. In what follows we shall use also the following notion. A root $\gamma' \in \Delta^+$ is called a *subroot* of $\gamma'' \in \Delta^+$ if $\gamma'' - \gamma' \neq 0$ may be expressed as a linear combination of simple roots with non-negative coefficients.

For any $\mathscr{G}$ it is enough to consider roots for which $n_k \neq 0$ for $1 \leqslant k \leqslant l$. Any other root β' may be considered as a root of a complex simple Lie algebra $\mathscr{G}'$ isomorphic to a subalgebra of $\mathscr{G}$ of rank $l' < l$, so that $\beta' = \sum_k n'_k \alpha'_k$ and $n'_k \neq 0$ for $1 \leqslant k \leqslant l'$ (α'_k being the simple roots of $\mathscr{G}'$). Thus in the case of the straight roots we shall consider always the case when $u = l-1$, and $\{i_1, \ldots, i_u, v\}$ will be a permutation of $\{1, \ldots, l\}$.

Let us have a Verma module V^λ over $U_q(\mathscr{G})$ as defined in the previous subsection. Let $\beta = n_1\alpha_1 + n_2\alpha_2 + \cdots + n_l\alpha_l$, where $n_k \in \mathbb{Z}_+$, be a straight root of the positive root system Δ^+ of $\mathscr{G}$, and m a positive integer. Let $\lambda \in \mathscr{H}^*$ be such that (2.2) is fulfilled with this choice of β and m, but is not fulfilled for any subroot of β. Then the singular vector of the Verma module V^λ corresponding to β and m is given by:

$$v_\lambda^{\beta,m} = \mathscr{P}_\lambda^{\beta,m} \otimes v_0 = \sum_{k_1=0}^{mn_{i_1}} \cdots \sum_{k_u=0}^{mn_{i_u}} c_{k_1 \cdots k_u} (X_{i_1}^-)^{mn_{i_1}-k_1} \cdots (X_{i_u}^-)^{mn_{i_u}-k_u}$$
$$\times (X_v^-)^{mn_v} (X_{i_u}^-)^{k_u} \cdots (X_{i_1}^-)^{k_1} \otimes v_0 , \tag{2.7a}$$

$$c_{k_1\cdots k_u} = (-1)^{k_1+\cdots+k_u} c_u \binom{mn_{i_1}}{k_1}_{q_{i_1}} \cdots \binom{mn_{i_u}}{k_u}_{q_{i_u}}$$

$$\times \frac{[(\lambda+\rho)(\tilde{H}_{i_1})]_{q_{i_1}}}{[(\lambda+\rho)(\tilde{H}_{i_1})-k_1]_{q_{i_1}}} \cdots \frac{[(\lambda+\rho)(\tilde{H}_{i_u})]_{q_{i_u}}}{[(\lambda+\rho)(\tilde{H}_{i_u})-k_u]_{q_{i_u}}}, \qquad (2.7\text{b})$$

where the indices $i_1, ..., i_u, v$ come from the representation (2.6), and $\tilde{H}_{i_1}\cdots\tilde{H}_{i_u}$ are linear combinations of the basis elements H_i of the Cartan subalgebra $\mathscr{H}$ of $\mathscr{G}$, which can be computed explicitly in all cases. This is presented below.

(1) $\mathscr{G}=A_l$, $(\alpha_i, \alpha_j)=-1$ for $|i-j|=1$, $(\alpha_i, \alpha_j)=2\delta_{ij}$ otherwise. Then every root $\beta\in\Delta^+$ is given by $\beta=\beta_{in}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_{i+n-1}$, where $1\leqslant i\leqslant l$, $1\leqslant n \leqslant l-i+1$. Note that *every root is straight* since

$$\beta_{i,n}=s_i(\beta_{i+1,n})=s_i\cdots s_{i+n-2}(\alpha_{i+n-1})=s_{i+n-1}\cdots s_{i+1}(\alpha_i)$$
$$=s_i\cdots s_{i+t-1}s_{i+n-1}\cdots s_{i+t+1}(\alpha_{i+t})$$
$$=s_{i+n-1}\cdots s_{i+t+1}s_i\cdots s_{i+t-1}(\alpha_{i+t}), \quad 0\leqslant t\leqslant n-1,$$

where we have demonstrated different forms of (2.6) in this case. For A_l the highest root [31] is given by $\tilde{\alpha}=\alpha_1+\alpha_2+\cdots+\alpha_l$.

Thus every root $\beta\in\Delta^+$ is the highest root of a subalgebra of A_l; explicitly β_{in} is the highest root of the subalgebra A_n with simple roots $\alpha_i, \alpha_{i+1}, ..., \alpha_{i+n-1}$. This means that it is enough to give the formula for the singular vector corresponding to the highest root. Thus in formula (2.7) with $\beta=\tilde{\alpha}$ we have $n_k=1$, $1\leqslant k\leqslant l$, and for the sets $i_1, ..., i_u, v$ we obtain from $\tilde{\alpha}=s_1s_2\cdots s_ts_ls_{l-1}\cdots s_{t+2}(\alpha_{t+1})$ the following:

$$\{i_1, ..., i_{l-1}; v\}=\{1, 2, ..., t, l, l-1, ..., t+2; t+1\}, \qquad (2.8)$$

$$\tilde{H}_{i_s}=\begin{cases} H^s, & 1\leqslant s\leqslant t, \\ H'^{l+t+1-s}, & t+1\leqslant s\leqslant j=l-1, \end{cases} \qquad (2.9)$$

$$H^k\equiv H_1+H_2+\cdots+H_k, \qquad H'^k\equiv H_l+H_{l-1}+\cdots+H_k. \qquad (2.10)$$

(2) $\mathscr{G}=D_l$, $l\geqslant 4$, $(\alpha_i, \alpha_j)=-1$ for $|i-j|=1$, $i,j\neq l$ and for $ij=l(l-2)$, $(\alpha_i, \alpha_j)=2\delta_{ij}$ otherwise. First we note that if $n_{l-2}+n_{l-1}+n_l\leqslant 2$ then the root β is a positive root of a subalgebra of D_l of type A_n, $n<l$. Thus it remains to consider straight roots $\beta_i\in\Delta^+$ given by $\beta_i=\alpha_i+\alpha_{i+1}+\cdots+\alpha_l$. Note that β_i is a root of the subalgebra D_{l-i+1} with simple roots $\alpha_i, \alpha_{i+1}, ..., \alpha_l$. This means that in order to account for all roots β_i it is enough to consider the root

$$\tilde{\beta}=\beta_1=\alpha_1+\alpha_2+\cdots+\alpha_l=s_1s_2\cdots s_{l-3}s_{l-1}s_l(\alpha_{l-2})$$
$$=s_l\cdots s_2(\alpha_1)=s_1s_2\cdots s_{l-3}s_{l-1}s_{l-2}(\alpha_l)$$
$$=s_1s_2\cdots s_{l-3}s_ls_{l-2}(\alpha_{l-1}).$$

Thus in formula (2.7) with $\beta=\tilde{\beta}$ we have $n_k=1$, $1\leqslant k\leqslant l$, and for the set $i_1, ..., i_u$, v we give only the values corresponding to the first representation of $\tilde{\beta}$ above, namely we have

$$\{i_1, ..., i_{l-1}; v\}=\{1, 2, ..., l-3, l-1, l; l-2\},$$

$$\tilde{H}_{i_s}=\begin{cases} H^s, & 1\leqslant s\leqslant l-3, \\ H_{s+1}, & s=l-2, l-1. \end{cases} \tag{2.11}$$

(3) $\mathscr{G}=E_l$, $l=6, 7, 8$, $(\alpha_i, \alpha_{i+1})=-1$, $i=1, ..., l-2$, $(\alpha_3, \alpha_l)=-1$, $(\alpha_i, \alpha_j)=2\delta_{ij}$ otherwise. First we note that if $n_2+n_4+n_l\leqslant 2$ then the root β is a positive root of a subalgebra of E_l of type A_n, $n<l$. Analogously, if $n_2+n_4+n_l=3$ and $n_1+n_5\leqslant 1$ the root β is a positive root of a subalgebra of E_l of type D_n, $n<l$. Thus it remains to consider the straight root

$$\begin{aligned}\tilde{\beta}&=\alpha_1+\cdots+\alpha_l=s_1s_2s_ls_{l-1}\cdots s_4(\alpha_3)\\ &=s_l\cdots s_2(a_1)=s_1s_2s_{l-1}\cdots s_4s_3(\alpha_l)\\ &=s_1s_2s_ls_3\cdots s_{l-2}(\alpha_{l-1}).\end{aligned}$$

Thus in formula (2.7) with $\beta=\tilde{\beta}$ we have $n_k=1$, $1\leqslant k\leqslant l$, and for the set $i_1, ..., i_u$, v we give only the values corresponding to the first representation of $\tilde{\beta}$ above, namely we have

$$\{i_1, ..., i_{l-1}; v\}=\{1, 2, l, l-1, ..., 4; 3\},$$

$$\tilde{H}_{i_s}=\begin{cases} H_s, & s=1, 2, \\ H_l, & s=3, \\ H''^{l+3-s}, & s=4, ..., l-1, \end{cases} \tag{2.12}$$

$$H''^k\equiv H_{l-1}+\cdots+H_k. \tag{2.13}$$

(4) $\mathscr{G}=B_l$, $l\geqslant 2$, $(\alpha_i, \alpha_j)=-2$ if $|i-j|=1$, $(\alpha_i, \alpha_j)=2\delta_{ij}(2-\delta_{il})$ otherwise. The straight roots are of two types: $\beta_{in}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_{i+n-1}$, $1\leqslant i\leqslant l$, $1\leqslant n\leqslant l-i+1$, and $\beta'_i=\alpha_i+\cdots+\alpha_{l-1}+2\alpha_l$, $1\leqslant i\leqslant l$. If $i+n-1<l$ then β_{in} is a positive root of a subalgebra of B_l of type A_n, $n<l$ (with the scalar products scaled by 2 and q replaced by q^2). Thus we are left with two types of straight roots $\beta_i=\beta_{i,l+1-i}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_l$, $1\leqslant i<l$, and β'_i. As above it is enough to account for the roots with $i=1$. Thus we consider

$$\begin{aligned}\tilde{\beta}&=\beta_1=\alpha_1+\cdots+\alpha_l=s_1\cdots s_{l-1}(\alpha_l),\\ \tilde{\beta}'&=\beta'_1=\alpha_1+\cdots+\alpha_{l-1}+2\alpha_l\\ &=s_1\cdots s_{l-2}s_l(\alpha_{l-1})[=s_l\cdots s_2(\alpha_1)].\end{aligned}$$

We note that

$$(\tilde\beta,\tilde\beta)=2\,,\qquad \tilde\beta^\vee=\tilde\beta=2\alpha_1^\vee+\dots+2\alpha_{l-1}^\vee+\alpha_l^\vee\,,$$
$$(\tilde\beta',\tilde\beta')=4\,,\qquad \tilde\beta'^\vee=\tfrac12\tilde\beta'=\alpha_1^\vee+\dots+\alpha_l^\vee\,.$$

Thus in formula (2.7) with $\beta=\tilde\beta$ we have $n_k=1$, $1\leqslant k\leqslant l$, and

$$\{i_1,\dots,i_{l-1};v\}=\{1,\dots,l-1;l\}\,,$$
$$\tilde H_{i_s}=H^s\,,\qquad q_{i_s}=q^2\,,\quad s=1,\dots,l-1\,;\tag{2.14}$$

while for $\beta=\tilde\beta'$ we have $n_k=1+\delta_{kl}$, $1\leqslant k\leqslant l$, and

$$\{i_1,\dots,i_{l-1};v\}=\{1,\dots,l-2,l;l-1\}\,,\tag{2.15a}$$
$$\tilde H_{i_s}=\begin{cases}H^s, & s=1,\dots,l-2\,,\\ H_l, & s=l-1\,,\end{cases}\qquad q_{i_s}=q^{2-\delta_{s\,l-1}}\,.\tag{2.15b}$$

(5) $\mathcal{G}=C_l$, $l\geqslant 3$ ($C_2\cong B_2$), $(\alpha_i,\alpha_j)=-1$ if $|i-j|=1$ and $i,j<l$, $(\alpha_i,\alpha_j)=-2$ if $ij=l(l-1)$, $(\alpha_i,\alpha_j)=2\delta_{ij}(1+\delta_{il})$ otherwise. The straight roots are of two types:

$$\beta_{in}=\alpha_i+\alpha_{i+1}+\dots+\alpha_{i+n-1}\,,\quad 1\leqslant i\leqslant l,\ 1\leqslant n\leqslant l-i+1\,,$$
$$\beta_i''=2\alpha_i+\dots+2\alpha_{l-1}+\alpha_l,\quad 1\leqslant i<l\,.$$

If $i+n-1<l$ then β_{in} is a positive root of a subalgebra of C_l of type A_n, $n<l$. Thus we are left with two types of straight roots $\beta_i=\beta_{i,l+1-i}=\alpha_i+\alpha_{i+1}+\dots+\alpha_l$, $1\leqslant i<l$, and β_i''. As above it is enough to account for the roots with $i=1$. Thus we consider

$$\tilde\beta=\beta_1=\alpha_1+\dots+\alpha_l=s_l\cdots s_2(\alpha_1)\,[=s_1\cdots s_{l-2}s_l(\alpha_{l-1})]\,,$$
$$\tilde\beta''=\beta_1''=2\alpha_1+\dots+2\alpha_{l-1}+\alpha_l=s_1\cdots s_{l-1}(\alpha_l)\,.$$

We note that

$$(\tilde\beta,\tilde\beta)=2\,,\qquad \tilde\beta^\vee=\tilde\beta=\alpha_1^\vee+\dots+\alpha_{l-1}^\vee+2\alpha_l^\vee\,,$$
$$(\tilde\beta'',\tilde\beta'')=4\,,\qquad \tilde\beta''^\vee=\tfrac12\tilde\beta''=\alpha_1^\vee+\dots+\alpha_l^\vee\,.$$

Thus in formula (2.7) with $\beta=\tilde\beta$ we have $n_k=1$, $1\leqslant k\leqslant l$, and

$$\{i_1,\dots,i_{l-1};v\}=\{l,\dots,2;1\}\,,$$
$$\tilde H_{i_s}=H'^{l+1-s}\,,\qquad q_{i_s}=q^{1+\delta_{s1}}\,,\quad s=1,\dots,l-1\,;\tag{2.16}$$

while for $\beta=\tilde\beta''$ we have $n_k=2-\delta_{kl}$, $1\leqslant k\leqslant l$, and

$$\{i_1,\dots,i_{l-1};v\}=\{1,\dots,l-1;l\}\,,$$
$$\tilde H_{i_s}=H^s\,,\qquad q_{i_s}=q\,,\quad s=1,\dots,l-1\,.\tag{2.17}$$

(6) $\mathcal{G}=F_4$, $(\alpha_1,\alpha_1)=(\alpha_2,\alpha_2)=2(\alpha_3,\alpha_3)=2(\alpha_4,\alpha_4)=4$ and $(\alpha_1,\alpha_2)=$

$(\alpha_2, \alpha_3) = 2(\alpha_3, \alpha_4) = -2$ are the non-zero products between the simple roots. We have straight roots of type A_2: $\alpha_1 + \alpha_2$, $\alpha_3 + \alpha_4$, of type B_2: $\alpha_2 + \alpha_3$, $\alpha_2 + 2\alpha_3$, of type B_3: $\alpha_1 + \alpha_2 + \alpha_3$, $\alpha_1 + \alpha_2 + 2\alpha_3$, of type C_3: $\alpha_2 + \alpha_3 + \alpha_4$, $\alpha_2 + 2\alpha_3 + 2\alpha_4$. Thus we are left with the two roots

$$\tilde{\beta} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = s_1 s_2 s_4(\alpha_3) ,$$
$$\tilde{\beta}' = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 = s_1 s_4 s_3(\alpha_2) .$$

We note that

$$(\tilde{\beta}, \tilde{\beta}) = 2 , \qquad \tilde{\beta}^\vee = \tilde{\beta} = 2\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee ,$$
$$(\tilde{\beta}'', \tilde{\beta}'') = 4 , \qquad \tilde{\beta}''^\vee = \tfrac{1}{2}\tilde{\beta}'' = \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee .$$

Thus in formula (2.7) with $\beta = \tilde{\beta}$ we have $n_k = 1$, $1 \leqslant k \leqslant 4$, and

$$\{i_1, ..., i_3; v\} = \{1, 2, 4; 3\} ,$$
$$\tilde{H}_{i_s} = \begin{cases} H^s, & s = 1, 2 , \\ H_4, & s = 3 , \end{cases} \qquad q_{i_s} = q^{2 - \delta_{s3}} ; \tag{2.18}$$

while for $\beta = \tilde{\beta}'$ we have $n_k = 1$, $k = 1, 2$, $n_k = 2$, $k = 3, 4$, and

$$\{i_1, ..., i_3; v\} = \{1, 4, 3; 2\} ,$$
$$\tilde{H}_{i_s} = \begin{cases} H_1, & s = 1 , \\ H'^{6-s}, & s = 2, 3 , \end{cases} \qquad q_{i_s} = q^{1 + \delta_{s1}} . \tag{2.19}$$

(7) $\mathscr{G} = G_2$, $(\alpha_1, \alpha_1) = 3(\alpha_2, \alpha_2) = -2(\alpha_1, \alpha_2) = 6$. The non-simple straight roots are the two roots

$$\tilde{\beta} = \alpha_1 + \alpha_2 = s_1(\alpha_2) , \qquad \tilde{\beta}''' = \alpha_1 + 3\alpha_2 = s_2(\alpha_1) .$$

We note that

$$(\tilde{\beta}, \tilde{\beta}) = 2 , \qquad \tilde{\beta}^\vee = \tilde{\beta} = 3\alpha_1^\vee + \alpha_2^\vee ,$$
$$(\tilde{\beta}''', \tilde{\beta}''') = 6 , \qquad \tilde{\beta}'''^\vee = \tfrac{1}{3}\tilde{\beta}''' = \alpha_1^\vee + \alpha_2^\vee .$$

Thus in formula (2.7) with $\beta = \tilde{\beta}$ we have $n_k = 1$, $k = 1, 2$, and

$$\{i_1; v\} = \{1; 2\} , \qquad \tilde{H}_{i_1} = H_1 , \qquad q_{i_1} = q^3 ; \tag{2.20}$$

while for $\beta = \tilde{\beta}'''$ we have $n_1 = 1$, $n_2 = 3$, and

$$\{i_1; v\} = \{2; 1\} , \qquad \tilde{H}_{i_1} = H_2 , \qquad q_{i_1} = q . \tag{2.21}$$

3. Duality

3.1. DUALITY BETWEEN HOPF ALGEBRAS

Two bialgebras $\mathcal{U}$, $\mathcal{A}$ are said to be *in duality* [A] if there exists a doubly non-degenerate bilinear form

$$\langle\ ,\ \rangle : \mathcal{U}\times\mathcal{A}\to\mathbb{C}\,, \qquad \langle\ ,\ \rangle : (u,a)\mapsto\langle u,a\rangle\,, \quad u\in\mathcal{U}, a\in\mathcal{A}\,, \tag{3.1}$$

such that, for u, $v\in\mathcal{U}$, a, $b\in\mathcal{A}$:

$$\langle u, ab\rangle = \langle\delta_{\mathcal{U}}(u), a\otimes b\rangle\,, \qquad \langle uv, a\rangle = \langle u\otimes v, \delta_{\mathcal{A}}(a)\rangle\,, \tag{3.2a}$$

$$\langle 1_{\mathcal{U}}, a\rangle = \epsilon_{\mathcal{A}}(a), \qquad \langle u, 1_{\mathcal{A}}\rangle = \epsilon_{\mathcal{U}}(u)\,. \tag{3.2b}$$

Two Hopf algebras $\mathcal{U}$, $\mathcal{A}$ are said to be *in duality* [A] if they are in duality as bialgebras and if

$$\langle\gamma_{\mathcal{U}}(u), a\rangle = \langle u, \gamma_{\mathcal{A}}(a)\rangle\,. \tag{3.2}$$

It is enough to define the pairing (3.1) between the generating elements of the two algebras. The pairing between any other elements of $\mathcal{U}$, $\mathcal{A}$ follows then from relations (3.2) and the standard bilinear form inherited by the tensor product. For example, suppose $\delta(u) = \sum_i u_i'\otimes u_i''$, then one has

$$\begin{aligned}\langle u, ab\rangle = \langle\delta_{\mathcal{U}}(u), a\otimes b\rangle &= \sum_i \langle u_i'\otimes u_i'', a\otimes b\rangle \\ &= \sum_i \langle u_i', a\rangle\langle u_i'', b\rangle\,.\end{aligned} \tag{3.3}$$

3.2. MATRIX QUANTUM GROUP $GL_{p,q}(2,\mathbb{C})$

In this subsection we review the two-parameter deformation of $GL(2,\mathbb{C})$ following ref. [DMMZ] (cf. also refs. [Ku, Su2, SWZ]). For more general multi-parametric deformations we refer to refs. [M2, Re2, FZ, Si].

Let p, $q\in\mathbb{C}\setminus\{0\}$. Consider next 2×2 matrices M with non-commuting matrix elements which perform linear transformations of $R_q(2|0)$ and $R_p(0|2)$, i.e.,

$$\{x_1', x_2'\}\in R_q(2|0)\,, \quad x_i' = M_{ij}x_j\,, \tag{3.4a}$$

$$\{\xi_1', \xi_2'\}\in R_p(0|2)\,, \quad \xi_i' = M_{ij}\xi_j\,, \tag{3.4b}$$

assuming that the elements of M commute with all x_i, ξ_i and summation over repeated indices is understood. Let us write the matrix M as

$$M = \begin{pmatrix} a & b \\ c & d\end{pmatrix}. \tag{3.5}$$

Then implementation of (3.4) gives that the matrix elements of M obey [DMMZ]:

$$ab=p^{-1/2}ba\,,\qquad ac=q^{-1/2}ca\,,$$
$$bd=q^{-1/2}db\,,\qquad cd=p^{-1/2}dc\,,$$
$$q^{1/2}bc=p^{1/2}cb\,,\qquad ad-da=(p^{-1/2}-q^{1/2})bc\,. \tag{3.6}$$

Let us denote by $A_{p,q}(2)$ the bialgebra generated by the matrix elements a, b, c, d with the following co-multiplication δ and co-unit ϵ [cf. also (1.21) for $n=2$],

$$\delta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=\begin{pmatrix} a\otimes a+b\otimes c & a\otimes b+b\otimes d \\ c\otimes a+d\otimes c & c\otimes b+d\otimes d \end{pmatrix}, \tag{3.7a}$$

$$\epsilon\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.7b}$$

Further, a quantum determinant $\det_{p,q}M\in A_{p,q}(2)$ is defined as follows:

$$\mathscr{D}\equiv\det_{p,q}M=ad-p^{-1/2}bc=ad-q^{-1/2}cb$$
$$=da-p^{1/2}cb=da-q^{1/2}bc\,, \tag{3.8}$$

and then we have [cf. (1.23)]

$$\delta(\mathscr{D})=\mathscr{D}\otimes\mathscr{D}\,,\qquad \epsilon(\mathscr{D})=1\,. \tag{3.9}$$

The crucial difference with the one-parameter case, which is obtained for $p=q$ (cf. subsection 1.3), is that the quantum determinant is not central but satisfies the following relations [DMMZ]:

$$[\mathscr{D},a]=[\mathscr{D},d]=0\,,\qquad p^{1/2}\mathscr{D}b=q^{1/2}b\mathscr{D}\,,\qquad q^{1/2}\mathscr{D}c=p^{1/2}c\mathscr{D}\,. \tag{3.10}$$

Further, if $\mathscr{D}\neq 0$ one extends the algebra by an element $\mathscr{D}^{-1}$ obeying

$$\mathscr{D}\mathscr{D}^{-1}=\mathscr{D}^{-1}\mathscr{D}=1_{\mathscr{A}}\,, \tag{3.11a}$$

from which follows [DMMZ]:

$$[\mathscr{D}^{-1},a]=0\,,\qquad [\mathscr{D}^{-1},d]=0\,,$$
$$q^{1/2}\mathscr{D}^{-1}b=p^{1/2}b\mathscr{D}^{-1}\,,\qquad p^{1/2}\mathscr{D}^{-1}c=q^{1/2}c\mathscr{D}^{-1}\,. \tag{3.11b}$$

Next one defines the left and right inverse matrix of M [DMMZ]:

$$M^{-1}=\mathscr{D}^{-1}\begin{pmatrix} d & -q^{1/2}b \\ -q^{-1/2}c & a \end{pmatrix}=\begin{pmatrix} d & -p^{1/2}b \\ -p^{-1/2}c & a \end{pmatrix}\mathscr{D}^{-1}\,. \tag{3.12}$$

Suppose that the bialgebra operations are defined on $\mathscr{D}^{-1}$. Then we have

$$\delta(\mathscr{D}^{-1})=\mathscr{D}^{-1}\otimes\mathscr{D}^{-1}\,,\qquad \epsilon(\mathscr{D}^{-1})=1\,. \tag{3.13}$$

The quantum group $GL_{p,q}(2,\mathbb{C})$ is defined as the Hopf algebra obtained from the bialgebra $A_{p,q}(2)$ extended by the element $\mathscr{D}^{-1}$ and with antipode given by the formula

$$\gamma(M)=M^{-1}\,. \tag{3.14}$$

From the above definition we have

$$\gamma(\mathscr{D})=\mathscr{D}^{-1}\,,\qquad \gamma(\mathscr{D}^{-1})=\mathscr{D}\,. \tag{3.15}$$

For $p=q$ one obtains from $\mathrm{GL}_{q,q}(2,\mathbb{C})$ the quantum groups $\mathrm{GL}_q(2,\mathbb{C})$ and $\mathrm{SL}_q(2,\mathbb{C})$, if the condition $\mathscr{D}\neq 0$, and $\mathscr{D}=1_{\mathscr{A}}$, respectively, holds.

3.3. DUALITY À LA SUDBERY FOR $\mathrm{GL}_{p,q}(2,\mathbb{C})$

In this subsection we review ref. [Do5], where we have applied to $A_{p,q}(2)$ the approach which Sudbery invented for $A_q(2)=A_{q,q}(2)$ [Su1].

For $A_{p,q}(2)$ we use the basis given by all monomials $f=f_{klmn}=a^k d^l b^m c^n$, where $k,l,m,n\in\mathbb{Z}_+$, and $f_{0000}=1_{\mathscr{A}}$. We postulate the following pairings for $f=a^k d^l b^m c^n$:

$$\langle A,g\rangle=k\delta_{m0}\delta_{n0}\,, \tag{3.16a}$$

$$\langle B,f\rangle=\delta_{m1}\delta_{n0}\,, \tag{3.16b}$$

$$\langle C,f\rangle=\delta_{m0}\delta_{n1}\,, \tag{3.16c}$$

$$\langle D,f\rangle=l\delta_{m0}\delta_{n0}\,. \tag{3.16d}$$

Let us denote by $\mathscr{U}_{p,q}$ the bialgebra dual to $A_{p,q}(2)$ and generated by A, B, C, D. Later we shall see that $\mathscr{U}_{p,q}$ has the structure of a Hopf algebra in duality with $\mathrm{GL}_{p,q}(2,\mathbb{C})$.

The following relations hold as consequences from (3.16):

$$\left\langle A,\begin{pmatrix}a&b\\c&d\end{pmatrix}\right\rangle=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad \left\langle D,\begin{pmatrix}a&b\\c&d\end{pmatrix}\right\rangle=\begin{pmatrix}0&0\\0&1\end{pmatrix}, \tag{3.17a}$$

$$\left\langle B,\begin{pmatrix}a&b\\c&d\end{pmatrix}\right\rangle=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad \left\langle C,\begin{pmatrix}a&b\\c&d\end{pmatrix}\right\rangle=\begin{pmatrix}0&0\\1&0\end{pmatrix}, \tag{3.17b}$$

$$\langle Y,1_{\mathscr{A}}\rangle=0\,,\quad Y=A,B,C,D\,, \tag{3.18}$$

$$\langle 1_{\mathscr{U}},a^k d^l b^m c^n\rangle=\delta_{m0}\delta_{n0}\,. \tag{3.19}$$

We would like to find the commutation relations between the generators of $\mathscr{U}_{p,q}$. First we obtain that the action on $f=a^k d^l b^m c^n$ of the monomials in $\mathscr{U}_{p,q}$ which are quadratic in the generators is given by the following:

$$\langle BC,f\rangle=\delta_{m0}\delta_{n0}\sum_{j=0}^{k-1}(pq)^{(j-l)/2}+q^{-1/2}\delta_{m1}\delta_{n1}\,, \tag{3.20a}$$

$$\langle CB,f\rangle = \delta_{m0}\delta_{n0}\sum_{j=0}^{l-1}(pq)^{-j/2} + p^{1/2}\delta_{m1}\delta_{n1} \,, \tag{3.20b}$$

$$\langle AB,f\rangle = (k+1)\delta_{m1}\delta_{n0} = (k+1)\langle B,f\rangle \,, \tag{3.21a}$$

$$\langle BA,f\rangle = k\delta_{m1}\delta_{n0} = k\langle B,f\rangle \,, \tag{3.21b}$$

$$\langle AC,f\rangle = k\delta_{m0}\delta_{n1} = k\langle C,f\rangle \,, \tag{3.22a}$$

$$\langle CA,f\rangle = (k+1)\delta_{m0}\delta_{n1} = (k+1)\langle C,f\rangle \,, \tag{3.22b}$$

$$\langle DB,f\rangle = l\delta_{m1}\delta_{n0} = l\langle B,f\rangle \,, \tag{3.23a}$$

$$\langle BD,f\rangle = (l+1)\delta_{m1}\delta_{n0} = (l+1)\langle B,f\rangle \,, \tag{3.23b}$$

$$\langle DC,f\rangle = (l+1)\delta_{m0}\delta_{n1} = (l+1)\langle C,f\rangle \,, \tag{3.24a}$$

$$\langle CD,f\rangle = l\delta_{m0}\delta_{n1} = l\langle C,f\rangle \,, \tag{3.24b}$$

$$\langle AD,f\rangle = \langle DA,f\rangle = kl\delta_{m0}\delta_{n0} = kl\langle 1_{\mathscr{U}},f\rangle \,. \tag{3.25}$$

Then we have:

$$q^{1/2}\langle BC, a^k d^l b^m c^n\rangle - p^{-1/2}\langle CB, a^k d^l b^m c^n\rangle = \frac{(pq)^{(l-k)/2}-1}{p^{1/2}-q^{-1/2}}\,\delta_{m0}\delta_{n0=} \,, \tag{3.26}$$

$$\langle [A,B],f\rangle = \langle B,f\rangle \,, \qquad \langle [A,C],f\rangle = -\langle C,f\rangle \,, \tag{3.27a}$$

$$\langle [D,B],f\rangle = -\langle B,f\rangle \,, \qquad \langle [D,C],f\rangle = \langle C,f\rangle \,, \tag{3.27b}$$

$$\langle [A,D],f\rangle = 0 \,. \tag{3.27c}$$

Note that relations (3.20)–(3.25) depend on the element f, but the commutation relations (3.27) do not. This is also true for (3.26); in order to see this we need the following formulae:

$$\langle A^s, a^k d^l b^m c^n\rangle = k^s\delta_{m0}\delta_{n0} \,, \quad s\in\mathbb{N} \,, \tag{3.28a}$$

$$\langle D^s, a^k d^l b^m c^n\rangle = l^s\delta_{m0}\delta_{n0} \,, \quad s\in\mathbb{N} \,, \tag{3.28b}$$

$$\langle r^A, a^k d^l b^m c^n\rangle = r^k\delta_{m0}\delta_{n0} \,, \quad r=p,q \,, \tag{3.29a}$$

$$\langle r^D, a^k d^l b^m c^n\rangle = r^l\delta_{m0}\delta_{n0} \,, \quad r=p,q \,, \tag{3.29b}$$

where we use the formal power series

$$r^Y = 1_{\mathscr{U}} + \sum_{k=1}^{\infty} Y^k(\ln r)^k/k!$$

Thus we obtain that the commutation relations in the algebra $\mathscr{U}_{p,q}$ are given by:

$$q^{1/2}BC-p^{-1/2}CB=\frac{(pq)^{(A-D)/2}-1_{\mathscr{U}}}{p^{1/2}-q^{-1/2}}, \tag{3.30}$$

$$[A, B]=B, \qquad [A, C]=-C,$$
$$[D, B]=-B, \qquad [D, C]=C, \qquad [A, D]=0. \tag{3.31}$$

Note that the generator $K=A+D$ commutes with all other generators of $\mathscr{U}_{p,q}$. Let us denote by $\mathscr{Z}$ the algebra spanned by K.

Next we are looking for the analogue of the splitting $U_q(\mathrm{sl}(2, \mathbb{C}))\otimes U_q(\mathscr{Z})$ which Sudbery [Su1] obtained in the one-parameter case. We try a similar change of basis:

$$H=A-D, \qquad \tilde{X}^{+}=q'^{-1/4}Bq'^{-H/4}, \qquad \tilde{X}^{-}=q'^{1/4}Cq'^{-H/4},$$
$$q'\equiv(pq)^{1/2}, \tag{3.32}$$

and we get the generators H, $\tilde{X}^{+}$, $\tilde{X}^{-}$ satisfy commutation relations (1.7) and (1.8) with $l=1$, $q_1=q\to q'$, $H_1=H$, $X_1^{\pm}=\tilde{X}^{\pm}$.

The factors $q'^{\pm 1/4}$ in (3.32) seem redundant, since factors $q'^{\pm\nu}$ for arbitrary $\nu\in\mathbb{C}$ will play the same role of the previous statement. Their significance becomes clear if we calculate the action of the new generators on $a^kd^lb^mc^n$, namely,

$$\langle H^s, a^kd^lb^mc^n\rangle=(k-l)^s\delta_{m0}\delta_{n0}, \qquad \langle q'^H, a^kd^lb^mc^n\rangle=q'^{k-l}\delta_{m0}\delta_{n0}, \tag{3.33a}$$

$$\langle K^s, a^kd^lb^mc^n\rangle=(k+l)^s\delta_{m0}\delta_{n0}, \qquad \langle q'^K, a^kd^lb^mc^n\rangle=q'^{k+l}\delta_{m0}\delta_{n0}, \tag{3.33b}$$

$$\langle \tilde{X}^{+}, a^kd^lb^mc^n\rangle=q'^{(l-k)/4}\delta_{m1}\delta_{n0}, \tag{3.33c}$$

$$\langle \tilde{X}^{-}, a^kd^lb^mc^n\rangle=q'^{(l-k)/4}\delta_{m0}\delta_{n1}. \tag{3.33d}$$

Remark. Thus the two parameters are glued together in the commutation relations and the action of the new basis of the algebra $\mathscr{U}_{p,q}$. This is in agreement with the general result of Drinfeld [D2] stating that the q-deformation of $U(\mathrm{sl}(2, \mathbb{C}))$ is unique. However, we shall see below that in the Hopf algebra relations the two parameters are not glued together, since in fact we are obtaining a deformation of $U((\mathrm{gl}(2, \mathbb{C}))$.

We turn now to the bialgebra structure of $\mathscr{U}_{p,q}$. The co-multiplication in the algebra $\mathscr{U}_{p,q}$ is given by

$$\delta_{\mathscr{U}}(A)=A\otimes 1_{\mathscr{U}}+q_{\mathscr{U}}\otimes A, \tag{3.34a}$$

$$\delta_{\mathscr{U}}(B)=B\otimes p^{A/2}q^{-D/2}+1_{\mathscr{U}}\otimes B, \tag{3.34b}$$

$$\delta_{\mathscr{U}}(C)=C\otimes q^{A/2}p^{-D/2}+1_{\mathscr{U}}\otimes C, \tag{3.34c}$$

$$\delta_{\mathscr{U}}(D)=D\otimes 1_{\mathscr{U}}+1_{\mathscr{U}}\otimes D, \tag{3.34d}$$

or in the new basis by

$$\delta_{\mathcal{U}}(H)=H\otimes 1_{\mathcal{U}}+1_{\mathcal{U}}\otimes H\,, \tag{3.35a}$$

$$\delta_{\mathcal{U}}(K)=K\otimes 1_{\mathcal{U}}+1_{\mathcal{U}}\otimes K\,, \tag{3.35b}$$

$$\delta_{\mathcal{U}}(\tilde{X}^+)=\tilde{X}^+\otimes(p/q)^{K/4}q'^{H/4}+q'^{-H/4}\otimes\tilde{X}^+\,, \tag{3.35c}$$

$$\delta_{\mathcal{U}}(\tilde{X}^-)=\tilde{X}^-\otimes(q/p)^{K/4}q'^{H/4}+q'^{-H/4}\otimes\tilde{X}^-\,. \tag{3.35d}$$

For the proof we use the duality property (3.2a), namely, we should have $\langle Y,f\rangle=\langle\delta_{\mathcal{U}}(Y),f_1\otimes f_2\rangle$, $Y=A, B, C, D$, for every splitting $f=f_1f_2$.

The co-unit relations in $\mathcal{U}_{p,q}$ are given by

$$\epsilon_{\mathcal{U}}(Y)=0\,,\qquad Y=A, B, C, D, H, K, \tilde{X}^{\pm}\,, \tag{3.36}$$

which follows from (3.18), (3.32) and $\langle u, 1_{\mathcal{A}}\rangle=\epsilon_{\mathcal{U}}(u)$ [cf. (3.2b)].

Let us assume now that $\mathcal{U}_{p,q}$ is a Hopf algebra in duality with $\mathrm{GL}_{p,q}(2,\mathbb{C})$. This assumption would be correct if we can define consistently the action of the generators of $\mathcal{U}_{p,q}$ on $\mathcal{D}^{-1}$ and an antipode in $\mathcal{U}_{p,q}$. We are even in a better situation since the action on $\mathcal{D}^{-1}$ and the antipode map in $\mathcal{U}_{p,q}$ are uniquely obtained as a consequence of the assumed duality. Namely, we have that the action of $\mathcal{U}_{p,q}$ on $\mathcal{D}^{-1}$ is given by

$$\langle 1_{\mathcal{U}}, \mathcal{D}^{-1}\rangle=1\,, \tag{3.37a}$$

$$\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}, \mathcal{D}^{-1}\right\rangle=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}. \tag{3.37b}$$

To prove (3.37a) we use (3.2b) and (3.13): $\langle 1_{\mathcal{U}}, \mathcal{D}^{-1}\rangle=\epsilon_{\mathcal{A}}(\mathcal{D}^{-1})=1$. For (3.37b) we use a corollary of (3.16):

$$\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}, \mathcal{D}\right\rangle=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{3.37c}$$

and also (3.18), (3.37a).

Next we obtain that the antipode map in $\mathcal{U}_{p,q}$ is given by

$$\gamma_{\mathcal{U}}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}-A & -B_p^{-A/2}q^{D/2}\\ -C_q^{-A/2}p^{D/2} & -D\end{pmatrix}. \tag{3.38}$$

Finally we can state the main result of ref. [Do5]:

The Hopf algebra $\mathcal{U}_{p,q}$ dual to $\mathrm{GL}_{p,q}(2,\mathbb{C})$ by relations (3.16) is isomorphic to $U_{(pq)^{1/2}}(\mathrm{sl}(2,\mathbb{C}))\otimes U_{p/q}(\mathcal{Z})$ as a commutation algebra, where $\mathcal{Z}$ is spanned by K, and $U_r(\mathcal{Z})$ is spanned by K, $r^{\pm K/4}$. The subalgebra $U_{p/q}(\mathcal{Z})$ is a Hopf subalgebra of $\mathcal{U}_{p,q}$, the commutation subalgebra generated by H, $\tilde{X}^{\pm}$ is not a Hopf subalgebra.

For $p=q$ the dual algebra to $\mathrm{GL}_q(2,\mathbb{C})$ is $U_q(\mathrm{sl}(2,\mathbb{C}))\otimes U(\mathcal{Z})$ as a tensor product of Hopf subalgebras. For $q=1$ the last statement reduces to the classical relation $U(\mathrm{gl}(2,\mathbb{C}))=U(\mathrm{sl}(2,\mathbb{C}))\otimes U(\mathscr{Z})$.

4. Real forms

4.1. OVERVIEW OF THE PROCEDURE

In this section we review and explain a canonical procedure proposed in ref. [Do6] for the q-deformation of the real forms $\mathscr{G}$ of complex Lie (super-)algebras associated with (generalized) Cartan matrices. Besides several examples in the text we may refer the reader to the lectures of, e.g., Lukierski, Nowicki and Ruegg, and also to refs. [Do6, Do7].

Let $\mathscr{G}$ be a real simple Lie algebra (below we shall need to extend the construction to real reductive Lie algebras). We shall use the standard deformation from subsection 2.2 for the simple components of the complexification $\mathscr{G}^{\mathbb{C}}$ of $\mathscr{G}$ to obtain the deformation $U_q(\mathscr{G})$ as a real form of $U_q(\mathscr{G}^{\mathbb{C}})$.

As in the undeformed case this means that there exists an antilinear (anti)involution σ of $U_q(\mathscr{G}^{\mathbb{C}})$ which preserves $U_q(\mathscr{G})$. Unlike the undeformed case it is necessary to consider both involutions and anti-involutions, since there are two possibilities for the deformation parameter q, i.e., either $|q|=1$ or $q\in\mathbb{R}$. For instance, $U_q(\mathrm{su}(2))$ has $|q|=1$ when σ is an involution and $q\in\mathbb{R}$ when σ is an anti-involution. Further, σ is a co-algebra (anti)homomorphism, i.e.,

$$\delta\circ\sigma=(\sigma\times\sigma)\circ\delta\,, \quad \text{or} \quad \delta\circ\sigma=(\sigma\times\sigma)\circ\delta'\,;$$

$$\epsilon(\sigma(X))=\bar{\epsilon}(X) \quad \forall X\in U_q(\mathscr{G}^{\mathbb{C}})\,.$$

Then the relations for the antipode are $\sigma\circ\gamma=\gamma\circ\sigma$ if σ is an algebra involution and a co-algebra homomorphism or if it is an algebra anti-involution and a co-algebra antihomomorphism and $(\sigma\circ\gamma)^2=\mathrm{id}$ otherwise [Sch]. One approach to the real forms would be to try to classify directly the possible conjugations σ.

Our approach is more constructive and the conjugations σ are obtained as a byproduct of the procedure proposed below. Though the procedure is described mostly in terms which are known from the undeformed case, we will stress which steps are necessitated by the q-deformation.

The first basic ingredient of our approach relies on the fact that the real forms $\mathscr{G}$ of a complex simple Lie algebra $\mathscr{G}^{\mathbb{C}}$ are in one-to-one correspondence with the Cartan automorphisms θ of $\mathscr{G}^{\mathbb{C}}$. This allows us to study the structure of the real forms and to find their explicit embeddings as real subalgebras of $\mathscr{G}^{\mathbb{C}}$ invariant under θ, and consequently, using the same generators, to find $U_q(\mathscr{G})$. This ingredient is enough for the compact case though we still have to specify the range of q.

The second basic ingredient of our procedure is related to the fact that a real non-compact simple Lie algebra has in general (a finite number of) non-conjugate Cartan subalgebras. For each choice of conjugacy class of Cartan subalgebras we get a different q-deformation.

The third basic ingredient is constituted by the Bruhat decompositions $\mathcal{G}=\mathcal{A}\oplus\mathcal{M}\oplus\tilde{\mathcal{N}}\oplus\mathcal{N}$, where $\mathcal{A}$ is a non-compact abelian sub-algebra, $\mathcal{M}$ (a reductive Lie algebra) is the centralizer of $\mathcal{A}$ in $\mathcal{G}$ (mod $\mathcal{A}$), and $\tilde{\mathcal{N}}$ and $\mathcal{N}$, are nilpotent subalgebras forming the positive and negative root spaces, respectively, of the root system ($\mathcal{G}$, $\mathcal{A}$). Consistently, the Cartan subalgebras of $\mathcal{G}$ have the decomposition $\mathcal{H}=\mathcal{A}\oplus\mathcal{H}^m$, where $\mathcal{H}^m$ is a Cartan subalgebra of $\mathcal{M}$. A general property of the deformation $U_q(\mathcal{G})$ obtained by our procedure is that $U_q(\mathcal{M})$, $U_q(\tilde{\mathcal{P}})$, $U_q(\mathcal{P})$ are Hopf subalgebras of $U_q(\mathcal{G})$, where $\mathcal{P}=\mathcal{A}\oplus\mathcal{M}\oplus\mathcal{N}$, $\tilde{\mathcal{P}}=\mathcal{A}\oplus\mathcal{M}\oplus\tilde{\mathcal{N}}$, are parabolic subalgebras of $\mathcal{G}$. (All notions are recalled below.)

Our exposition is organized as follows. We fix a real simple Lie algebra $\mathcal{G}$ and its most non-compact Cartan subalgebra $\mathcal{H}_0$; then we present the procedure since it is most simple in this case. Then we point out the modifications necessary in order to consider Cartan subalgebras $\mathcal{H}$ of $\mathcal{G}$ which are non-conjugate to $\mathcal{H}_0$. Until this moment we consider only the so-called minimal parabolic subalgebras $\mathcal{P}_0$ (which are different for non-conjugate Cartan subalgebras). Next, for an arbitrary Cartan subalgebra, we extend the procedure for arbitrary parabolic subalgebras. Finally we note that we need to generalize the whole procedure to reductive Lie algebras, which is straightforward.

4.2. q-DEFORMATION WITH THE MOST NON-COMPACT CARTAN SUBALGEBRA

First we recall some standard facts on real semisimple Lie algebras. The basic reference for that is ref. [B]. Let $\mathcal{G}$ be a real semisimple Lie algebra and $\mathcal{K}$ be the maximal compact subalgebra of $\mathcal{G}$. Then $\mathcal{G}=\mathcal{K}\oplus\mathcal{P}$ is the Cartan decomposition of $\mathcal{G}$, the subspace $\mathcal{P}$ is non-compact. Let θ be the Cartan involution in $\mathcal{G}$ so that $\theta X=X$, $X\in\mathcal{K}$, $\theta X=-X$, $X\in\mathcal{P}$.

Let $\mathcal{A}_0$ be the maximal subspace of $\mathcal{P}$ which is an abelian subalgebra of $\mathcal{G}$; all non-compact abelian subalgebras of $\mathcal{G}$ with maximal dimension are conjugate to $\mathcal{A}_0$; $r_0=\dim\mathcal{A}$ is the *real* (or *split*) rank of $\mathcal{G}$. For different $\mathcal{G}$ the real rank r_0 may vary from 0 (then $\mathcal{G}$ is compact), up to $l=\text{rank}\,\mathcal{G}^{\mathbb{C}}$ [then $\mathcal{G}$ is called maximally split, e.g., $\text{sl}(n,\mathbb{R})$, $\text{so}(n,n)$, $\text{so}(n+1,n)$, $\text{sp}(n,\mathbb{R})$].

Let Δ_{R}^0 be the root system of the pair ($\mathcal{G}$, $\mathcal{A}_0$), also called ($\mathcal{A}_0$-)*restricted root system*:

$$\Delta_{\mathrm{R}}^0=\{\lambda\in\mathcal{A}_0^*\,|\,\lambda\neq0,\ \mathcal{G}_\lambda\neq0\}\,,$$

$$\mathcal{G}_\lambda=\{X\in\mathcal{G}\,|\,[Y,X]=\lambda(Y)X,\quad\forall\,Y\in\mathcal{A}_0\}\,. \tag{4.1}$$

The elements of $\Delta_{\mathrm{R}}^0=\Delta_{\mathrm{R}}^{0+}\cup\Delta_{\mathrm{R}}^{0-}$ are called ($\mathcal{A}_0$-)*restricted roots*; if $\lambda\in\Delta_{\mathrm{R}}^0$, $\mathcal{G}_\lambda$ are called ($\mathscr{A}_0$-)*restricted root spaces*, $\dim_{\mathbb{R}} \mathscr{G}_\lambda \geqslant 1$. Now we can introduce the subalgebras corresponding to the positive ($\Delta_{\mathbb{R}}^{0+}$) and negative ($\Delta_{\mathbb{R}}^{0-}$) restricted roots:

$$\tilde{\mathscr{N}}_0 = \bigoplus_{\lambda \in \Delta_{\mathbb{R}}^{0+}} \mathscr{G}_\lambda = \tilde{\mathscr{N}}_0^1 \oplus \tilde{\mathscr{N}}_0^2 ,$$

$$\mathscr{N}_0 = \bigoplus_{\lambda \in \Delta_{\mathbb{R}}^{0-}} \mathscr{G}_\lambda = \mathscr{N}_0^1 \oplus \mathscr{N}_0^2 = \theta \tilde{\mathscr{N}}_0 , \qquad (4.2)$$

where $\tilde{\mathscr{N}}_0^1$ and $\tilde{\mathscr{N}}_0^2$ are the direct sum of $\mathscr{G}_\lambda$ with $\dim_{\mathbb{R}} \mathscr{G}_\lambda = 1$ and $\dim_{\mathbb{R}} \mathscr{G}_\lambda > 1$, respectively, and analogously for $\mathscr{N}_0^a = \theta \tilde{\mathscr{N}}_0^a$. Then we have the (Bruhat) decompositions which we shall use for our q-deformation:

$$\mathscr{G} = \tilde{\mathscr{N}}_0 \oplus \mathscr{A}_0 \oplus \mathscr{M}_0 \oplus \mathscr{N}_0 = \tilde{\mathscr{N}}_0^1 \oplus \tilde{\mathscr{N}}_0^2 \oplus \mathscr{A}_0 \oplus \mathscr{M}_0 \oplus \mathscr{N}_0^1 \oplus \mathscr{N}_0^2 , \qquad (4.3)$$

where $\mathscr{M}_0$ is the centralizer of $\mathscr{A}_0$ in $\mathscr{K}$, i.e., $\mathscr{M}_0 = \{X \in \mathscr{K} \mid [X, Y] = 0, \ \forall Y \in \mathscr{A}_0\}$. In general $\mathscr{M}_0$ is a compact reductive Lie algebra, and we shall write $\mathscr{M}_0 = \mathscr{M}_0^{\mathrm{s}} \oplus \mathscr{Z}_0^m$, where $\mathscr{M}_0^{\mathrm{s}} = [\mathscr{M}_0, \mathscr{M}_0]$ is the semisimple part of $\mathscr{M}_0$, and $\mathscr{Z}_0^m$ is the centre of $\mathscr{M}_0$. Note that $\tilde{\mathscr{P}}_0^0 \equiv \tilde{\mathscr{N}}_0 \oplus \mathscr{A}_0 \oplus \mathscr{M}_0$, $\mathscr{P}_0^0 \equiv \mathscr{A}_0 \oplus \mathscr{M}_0 \oplus \mathscr{N}_0$ are subalgebras of $\mathscr{G}$, the so-called minimal parabolic subalgebras of $\mathscr{G}$ for that choice of Cartan subalgebra.

Further let $\mathscr{H}_0^m$ be the Cartan subalgebra of $\mathscr{M}_0$, i.e., $\mathscr{H}_0^m = \mathscr{H}_0^{ms} \oplus \mathscr{Z}_0^m$, where $\mathscr{H}_0^{ms}$ is the Cartan subalgebra of $\mathscr{M}_0^{\mathrm{s}}$. Then $\mathscr{H}_0 \equiv \mathscr{H}_0^m \oplus \mathscr{A}_0$ is a Cartan subalgebra of $\mathscr{G}$, the most non-compact one; $\dim_{\mathbb{R}} \mathscr{H}_0 = \dim_{\mathbb{R}} \mathscr{H}_0^{ms} + \dim_{\mathbb{R}} \mathscr{Z}_0^m + r_0$. We choose $\mathscr{H}_0$ to be also the Cartan subalgebra of $U_q(\mathscr{G})$. Let $\mathscr{H}^{\mathbb{C}}$ be the complexification of $\mathscr{H}_0$ ($l = \mathrm{rank}\ \mathscr{G}^{\mathbb{C}} = \dim_{\mathbb{C}} \mathscr{H}^{\mathbb{C}}$); then it is a Cartan subalgebra of $\mathscr{G}^{\mathbb{C}}$ and $U_q(\mathscr{G}^{\mathbb{C}})$.

It is important for our procedure to choose consistently the basis of the rest of $\mathscr{G}$ and $\mathscr{G}^{\mathbb{C}}$, and thus of $U_q(\mathscr{G})$. For this we use the classification of the roots from Δ with respect to $\mathscr{H}_0$. The set

$$\Delta_{\mathrm{r}}^0 \equiv \{\alpha \in \Delta \mid \alpha|_{\mathscr{H}_0^m} = 0\}$$

is called the set of *real roots,*

$$\Delta_{\mathrm{i}}^0 \equiv \{\alpha \in \Delta \mid \alpha|_{\mathscr{A}_0} = 0\}$$

the set of *imaginary roots,* and $\Delta_{\mathrm{c}}^0 \equiv \Delta \setminus (\Delta_{\mathrm{r}}^0 \cup \Delta_{\mathrm{i}}^0)$ the set of *complex roots.* Thus $\Delta = \Delta_{\mathrm{r}}^0 \cup \Delta_{\mathrm{i}}^0 \cup \Delta_{\mathrm{c}}^0$. Further, let $\alpha \in \Delta^+$, let $\mathscr{L}_\alpha^{\mathrm{c}}$ be the complex linear span of H_α, X_α, $X_{-\alpha}$, and let $\mathscr{L}_\alpha = \mathscr{L}_\alpha^{\mathrm{c}} \cap \mathscr{G}$. Then $\dim_{\mathbb{R}} \mathscr{L}_\alpha = 3$ iff $\alpha \in \Delta_{\mathrm{r}}^0 \cup \Delta_{\mathrm{i}}^0$. If $\alpha \in \Delta_{\mathrm{r}}^0$ then $X_\alpha \in \mathscr{P}^{\mathbb{C}}$ and $\mathscr{L}_\alpha$ is non-compact. Since the Cartan subalgebra is $\mathscr{H}_0$, then $X_\alpha \in \mathscr{K}^{\mathbb{C}}$ and $\mathscr{L}_\alpha$ is compact if $\alpha \in \Delta_{\mathrm{i}}^0$. The algebras $\mathscr{L}_\alpha$ are given by

$$\mathscr{L}_\alpha = \mathrm{rls}\{H_\alpha, X_\alpha, X_{-\alpha}\} , \quad \alpha \in \Delta_{\mathrm{r}}^{0+} , \qquad (4.4a)$$

$$\mathscr{L}_\alpha = \mathrm{rls}\{\mathrm{i}H_\alpha, X_\alpha - X_{-\alpha}, \mathrm{i}(X_\alpha + X_{-\alpha})\} , \quad \alpha \in \Delta_{\mathrm{i}}^{0+} , \qquad (4.4b)$$

where rls stands for real linear span.

Note that there is a one-to-one correspondence between the real roots $\alpha \in \Delta_{\mathrm{r}}^0$ and the restricted roots $\lambda \in \Delta_{\mathbb{R}}^0$ with $\dim_{\mathbb{R}} \mathscr{G}_\lambda = 1$ and naturally this correspondence is realized by the restriction $\lambda = \alpha|_{\mathscr{A}_0}$. Thus we take the elements in (4.4a), $X_\alpha^\pm$ for $\alpha \in \Delta_{\mathrm{r}}^0$, also as elements of $U_q(\mathscr{G})$. These generators obey (1.11) and if $\alpha \in \Delta_{\mathrm{r}}^{0+} \cap \Delta_{\mathrm{S}}$ also (1.14), and otherwise as explained after (1.14).

In particular, formulae (4.4a) determine completely a q-deformation of any maximally split real form (or normal real form), when all roots are real, $\mathcal{M}_0=0$, $\mathcal{H}_0=\mathcal{A}_0$, and (4.3) is reduced to

$$\mathcal{G}=\tilde{\mathcal{N}}_0\oplus\mathcal{A}_0\oplus\mathcal{N}_0\,, \tag{4.5}$$

i.e., this is the restriction to $\mathbb{R}$ of the standard decomposition $\mathcal{G}^{\mathbb{C}}=\mathcal{G}^{\mathbb{C}}_+\oplus\mathcal{H}^{\mathbb{C}}\oplus\mathcal{G}^{\mathbb{C}}_-$, and hence $U_q(\mathcal{G})$ is just the restriction of $U_q(\mathcal{G}^{\mathbb{C}})$ to $\mathbb{R}$ with $q\in\mathbb{R}$. Thus we also inherit the property that $U_q(\tilde{\mathcal{N}}_0\oplus\mathcal{A}_0)$ and $U_q(\mathcal{N}_0\oplus\mathcal{A}_0)$ are Hopf subalgebras of $U_q(\mathcal{G})$, since $U_q(\mathcal{G}^{\mathbb{C}}_\pm\oplus\mathcal{H}^{\mathbb{C}})$ are Hopf subalgebras of $U_q(\mathcal{G}^{\mathbb{C}})$. Note that σ here is an antilinear involution and co-algebra homomorphism such that $\sigma(Y)=Y$, $\forall Y\in U_q(\mathcal{G}^{\mathbb{C}})$. For the classical complex Lie algebras these forms are $U_q(\mathrm{sl}(n,\mathbb{R}))$, $U_q(\mathrm{so}(n,n))$, $U_q(\mathrm{so}(n+1,n))$, $U_q(\mathrm{sp}(n,\mathbb{R}))$, which are dual to the matrix quantum groups $\mathrm{SL}_q(n,\mathbb{R})$, $\mathrm{SO}_q(n,n)$, $\mathrm{SO}_q(n,n+1)$, $\mathrm{Sp}_q(n,\mathbb{R})$, introduced in ref. [FRT] from a different point of view than ours.

Further note that the set of the imaginary roots Δ^0_{i} may be identified with the root system of $\mathcal{M}^{\mathrm{s}\mathbb{C}}_0$. Thus the elements in (4.4b) give the Hopf algebra $U_q(\mathcal{M}^{\mathrm{s}}_0)$ by the formulae:

$$[C^+_\alpha, C^-_\alpha]=\frac{\sinh(\tilde{\mathcal{H}}_\alpha h_\alpha/2)}{\sin(h_\alpha/2)}\,, \qquad [\tilde{H}_\alpha, C^\pm_\alpha]=\pm C^\mp_\alpha\,,$$

$$q_\alpha=q^{(\alpha,\alpha)/2}=\mathrm{e}^{-\mathrm{i}h_\alpha}\,, \tag{4.6a}$$

$$C^+_\alpha=(\mathrm{i}/\sqrt{2})(X_\alpha+X_{-\alpha})\,, \qquad C^-_\alpha=(1/\sqrt{2})(X_\alpha-X_{-\alpha})\,,$$

$$\tilde{H}_\alpha=-\mathrm{i}H_\alpha\,, \tag{4.6b}$$

$$\delta(C^\pm_\alpha)=C^\pm_\alpha\otimes\mathrm{e}^{\tilde{H}_\alpha h_\alpha/4}+\mathrm{e}^{-\tilde{H}_\alpha h_\alpha/4}\otimes C^\pm_\alpha\,, \quad \alpha\in\Delta^{0+}_{\mathrm{i}}\cap\Delta_{\mathrm{S}}\,. \tag{4.6c}$$

Since $\mathcal{M}_0=\mathcal{M}^{\mathrm{s}}_0\oplus\mathcal{Z}^m_0$ is a compact reductive Lie algebra we have to choose how to do the deformation in such cases. Our choice is to preserve the reductive structure, i.e., writing in more detail

$$\mathcal{M}_0=\bigoplus_j \mathcal{M}^{\mathrm{s}j}_0\oplus\bigoplus_k \mathcal{Z}^{mk}_0\,,$$

where $\mathcal{M}^{\mathrm{s}j}_0$ is simple and $\mathcal{Z}^{mk}_0$ is one dimensional; then we shall have the Hopf algebra

$$U_q(\mathcal{M}_0)=\bigotimes_j U_q(\mathcal{M}^{\mathrm{s}j}_0)\otimes\bigotimes_k U_q(\mathcal{Z}^{mk}_0)\,,$$

where we also have to specify that if $\mathcal{Z}^{mk}_0$ is spanned by K, then $U_q(\mathcal{Z}^{mk}_0)$ is spanned by K, $q^{\pm K/4}$.

In particular, formulae (4.6) (with $h_\alpha\in\mathbb{R}$) determine completely the unique q-deformation of any compact simple Lie algebra (when all roots of Δ are imaginary). Here one may take σ as an antilinear involution and co-algebra homomorphism such that $\sigma(X^\pm_\alpha)=-X^\mp_\alpha$, $\forall\alpha\in\Delta$, $\sigma(H)=-H$, $\forall H\in\mathcal{H}$. Note that in this case the q-deformation inherited from $U_q(\mathscr{G}^{\mathrm{C}})$ is often used in the physics literature without the basis change (4.6b).

Returning to the general situation, so far we have chosen consistently the generators of $\tilde{\mathscr{N}}_0^1 \oplus \mathscr{A}_0 \oplus \mathscr{M}_0 \oplus \mathscr{N}_0^1$ [cf. (4.2)] as linear combinations of the generators of

$$\mathscr{H}_0 \oplus \bigoplus_{\alpha \in \Delta_{\mathrm{r}}^0 \cup \Delta_{\mathrm{i}}^0} \mathscr{G}_\alpha .$$

Now it remains to choose consistently the generators of $\tilde{\mathscr{N}}_0^2$ and $\mathscr{N}_0^2$ as linear combinations of the generators of the rest of $\mathscr{G}^{\mathrm{C}}$, i.e., of $\bigoplus_{\alpha \in \Delta_{\mathrm{c}}^{0+}} \mathscr{G}_\alpha$ and $\bigoplus_{\alpha \in \Delta_{\mathrm{c}}^{0-}} \mathscr{G}_\alpha$, respectively. If $\alpha \in \Delta_{\mathrm{c}}^0$, $\lambda = \alpha|_{\mathscr{A}_0}$, then $\dim_{\mathrm{R}} \mathscr{G}_\lambda > 1$. Let $\Delta_\lambda = \{\alpha \in \Delta | \alpha|_{\mathscr{A}_0} = \lambda\}$. If $\alpha \in \Delta_{\mathrm{c}}^0$, then we have $X_\alpha = Y_\alpha + Z_\alpha$, where $Y_\alpha \in \mathscr{P}^{\mathrm{C}}$, $Z_\alpha \in \mathscr{K}^{\mathrm{C}}$. Now we can see that $\mathscr{G}_\lambda = \mathrm{rls}\{\tilde{X}_\alpha = Y_\alpha + \mathrm{i}Z_\alpha, \forall \alpha \in \Delta_\lambda\}$. The actual choice of basis in $\mathscr{G}_\lambda$ is a matter of convenience and is related to the choice of σ and q, and to the general property that $U_q(\tilde{\mathscr{P}}_0^0)$ and $U_q(\mathscr{P}_0^0)$ are Hopf subalgebras of $U_q(\mathscr{G})$.

4.3. q-DEFORMATIONS WITH ARBITRARY CARTAN SUBALGEBRAS

For the purposes of q-deformations we need also to discuss Cartan subalgebras $\mathscr{H}$ which are not conjugate to $\mathscr{H}_0$. Cartan subalgebras which represent different conjugacy classes may be chosen as $\mathscr{H} = \mathscr{H}_k \oplus \mathscr{A}$, where $\mathscr{H}_k$ is compact, $\mathscr{A}$ is non-compact, $\dim \mathscr{A} < \dim \mathscr{A}_0$ if $\mathscr{H}$ is non-conjugate to $\mathscr{H}_0$. [The dimensions of $\mathscr{H}_k$ and $\mathscr{A}$ may vary from 0 to $\dim \mathscr{H}$ even for a fixed $\mathscr{G}$; e.g., for $\mathrm{sp}(2n, 2n)$ one has $\mathscr{H}_0 = \mathscr{A}_0$, $\mathscr{H}_{0k} = 0$; however, there exists a compact Cartan subalgebra with $\mathscr{H} \cong \mathscr{H}_k$, $\mathscr{A} = 0$.]

All notions introduced until now are easily generalized for $\mathscr{H} = \mathscr{H}_k \oplus \mathscr{A}$ non-conjugate to $\mathscr{H}_0$. We note the differences and notationwise we drop all 0 subscripts and superscripts. One difference is that the algebra $\mathscr{M}$ is the centralizer of $\mathscr{A}$ in $\mathscr{G}$ (mod $\mathscr{A}$) and thus is in general a non-compact reductive Lie algebra which has the compact $\mathscr{H}_k$ as Cartan subalgebra (besides, in general, other non-compact Cartan subalgebras); in particular, if $\mathscr{G}$ has a compact Cartan subalgebra then for the choice $\mathscr{A} = 0$ one has $\mathscr{M} = \mathscr{G}$. For the purposes of the q-deformation we shall use this compact Cartan subalgebra, i.e., we set $\mathscr{H}^m = \mathscr{H}_k$. Further, the classification of the roots of Δ with respect to $\mathscr{H}$ goes as before. The difference is that if $\alpha \in \Delta_{\mathrm{i}}$ then $\mathscr{L}_\alpha$ may be also non-compact. Thus for $\alpha \in \Delta_{\mathrm{i}}$ the root α is called *singular*, $\alpha \in \Delta_{\mathrm{s}}$, if $\mathscr{L}_\alpha$ is non-compact, and α is called *compact*, $\alpha \in \Delta_{\mathrm{k}}$, if $\mathscr{L}_\alpha$ is compact. Thus $\Delta_{\mathrm{i}} = \Delta_{\mathrm{s}} \cup \Delta_{\mathrm{k}}$. Formulae (4.4b) hold for Δ_{k}, while for $\alpha \in \Delta_{\mathrm{s}}$ we have

$$\mathscr{L}_\alpha = \mathrm{rls}\{\mathrm{i}H_\alpha, \mathrm{i}(X_\alpha - X_{-\alpha}), X_\alpha + X_{-\alpha}\}, \quad \alpha \in \Delta_{\mathrm{s}}^+ , \tag{4.7a}$$

$$[S_\alpha^+, S_\alpha^-] = \frac{\sinh(\tilde{H}_\alpha h_\alpha/2)}{\sin(h_\alpha/2)}, \qquad [\tilde{H}_\alpha, S_\alpha^\pm] = \mp S_\alpha^\mp ,$$

$$q_\alpha = q^{(\alpha,\alpha)/2} = \mathrm{e}^{-\mathrm{i}h_\alpha}, \tag{4.7b}$$

$$S^+_\alpha = (1/\sqrt{2})(X_\alpha + X_{-\alpha}), \qquad S^-_\alpha = (i/\sqrt{2})(X_\alpha - X_{-\alpha}),$$

$$\tilde{H}_\alpha = -iH_\alpha, \tag{4.7c}$$

$$\delta(S^\pm_\alpha) = S^\pm_\alpha \otimes e^{\tilde{H}_\alpha h_\alpha/4} + e^{-\tilde{H}_\alpha h_\alpha/4} \otimes S^\pm_\alpha, \quad \alpha \in \Delta^+_s \cap \Delta_S. \tag{4.7d}$$

Further as before the set of the imaginary roots in Δ may be identified with the root system of $\mathcal{M}^{s\mathbb{C}}$. Thus formulae (4.6) and (4.7) give also the deformation $U_q(\mathcal{M}^s)$. Since the centre of $\mathcal{M}$ is compact (it is in the Cartan subalgebra $\mathcal{H}^m$ which is compact) then the deformation $U_q(\mathcal{Z}^m)$ is given as after (4.6). Thus the Hopf algebra $U_q(\mathcal{M})$ is given. Otherwise, the considerations for the factors $\mathcal{N}$ and $\tilde{\mathcal{N}}$ go as before.

4.4. q-DEFORMATIONS FOR ARBITRARY PARABOLIC SUBALGEBRAS AND REDUCTIVE LIE (SUPER-)ALGEBRAS

Until now our data are the non-conjugate Cartan subalgebras $\mathcal{H} = \mathcal{H}_k \oplus \mathcal{A}$ and the related Bruhat decompositions:

$$\mathcal{G} = \tilde{\mathcal{N}} \oplus \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{N} = \tilde{\mathcal{N}}^1 \oplus \tilde{\mathcal{N}}^2 \oplus \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{N}^1 \oplus \mathcal{N}^2. \tag{4.8}$$

In this decomposition a special role in the q-deformations is played by the subalgebra $\mathcal{P}_0 = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ (or equivalently by its Cartan involution conjugate $\tilde{\mathcal{P}}_0 = \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$). It is called a minimal parabolic subalgebra. A standard parabolic subalgebra is any subalgebra $\mathcal{P}'$ of $\mathcal{G}$ such that $\mathcal{P}_0 \subseteq \mathcal{P}'$. The number of standard parabolic subalgebras, including $\mathcal{P}_0$ and $\mathcal{G}$, is 2^r, $r = \dim \mathcal{A}$. They are all of the form $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, $\mathcal{M}' \supseteq \mathcal{M}$, $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{N}' \subseteq \mathcal{N}$; $\mathcal{M}'$ is the centralizer of $\mathcal{A}'$ in $\mathcal{G}$ (mod $\mathcal{A}'$); $\mathcal{N}'$ ($\tilde{\mathcal{N}}' = \theta \mathcal{N}'$) is comprised of the negative (positive) root spaces of the restricted root system Δ'_R of ($\mathcal{G}$, $\mathcal{A}'$). One also has the analogue of (4.2), (4.8):

$$\mathcal{G} = \tilde{\mathcal{N}}' \oplus \mathcal{A}' \oplus \mathcal{M}' \oplus \mathcal{N}'. \tag{4.9}$$

Note that $\mathcal{M}'$ is a non-compact reductive Lie algebra which has a non-compact Cartan subalgebra $\mathcal{H}'^m \cong \mathcal{H}_k \oplus \mathcal{H}_n$, where $\mathcal{H}_n$ is non-compact and $\mathcal{A} \cong \mathcal{H}_n \oplus \mathcal{A}'$. This Cartan subalgebra $\mathcal{H}'^m$ of $\mathcal{M}'$ will be chosen for the purposes of the q-deformation.

Thus we need to extend our scheme to non-compact reductive Lie algebras. Let $\hat{\mathcal{G}} = \mathcal{G} \oplus \mathcal{Z} = \hat{\mathcal{K}} \oplus \hat{\mathcal{P}}$ be a real reductive Lie algebra, where $\mathcal{G}$ is the semisimple part of $\hat{\mathcal{G}}$, $\mathcal{Z}$ is the centre of $\hat{\mathcal{G}}$; $\hat{\mathcal{K}}$, $\hat{\mathcal{P}}$ are the $+1$, -1 eigenspaces of the Cartan involution $\hat{\theta}$; $\hat{\mathcal{A}}' = \mathcal{A}' \oplus \mathcal{Z}_p$ is the analogue of $\mathcal{A}'$, $\mathcal{Z}_p = \mathcal{Z} \cap \hat{\mathcal{P}}$. The root system of the pair ($\hat{\mathcal{G}}$, $\hat{\mathcal{A}}'$) coincides with Δ'_R and the subalgebras $\tilde{\mathcal{N}}'$ and $\mathcal{N}'$ are inherited from $\mathcal{G}$. The decomposition (4.6) then is

$$\hat{\mathcal{G}} = \tilde{\mathcal{N}}' \oplus \hat{\mathcal{A}}' \oplus \hat{\mathcal{M}}' \oplus \mathcal{N}', \tag{4.10}$$

where $\hat{\mathcal{M}}' = \mathcal{M}'^s \oplus \hat{\mathcal{Z}}'^m$, $\hat{\mathcal{Z}}'^m = \mathcal{Z}'^m \oplus \mathcal{Z} \cap \hat{\mathcal{K}}$. As in the compact reductive case we choose a deformation which preserves the splitting of $\hat{\mathcal{G}}$, i.e., $U_q(\hat{\mathcal{G}}) =$

$U_q(\mathcal{G}) \otimes U_q(\mathcal{Z})$, and even further into simple Lie subalgebras and one-dimensional central subalgebras.

Let us stress again the general property of the deformation $U_q(\mathcal{G})$ obtained by the above procedure, which is that $U_q(\mathcal{M}_0)$, $U_q(\tilde{\mathcal{P}}_0)$, $U_q(\mathcal{P}_0)$ are Hopf subalgebras of $U_q(\mathcal{G})$.

All notions above are easily generalized to the real forms of the basic classical Lie superalgebras [K], and thus our approach is immediately generalized to the deformation of such superalgebras [Do6].

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